COMMUNICATION CIRCUITS

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PREFACE TO THE THIRD EDITION

The theory of communication circuits as presented in this textbook is intended as first-course material for all students of communication engineering, regardless of the frequency range with which they will be concerned. The basic principles of communication transmission lines and their associated networks are presented, covering the frequency range from voice frequencies through the very high frequencies which are coming into use today. Chapters XII, XIII, and XIV on Wave Guides and XV on Coaxial Cables apply entirely to the microwave region. Throughout the text, wherever the problem at hand could be appropriately applied to the high-frequency range this was done.

Owing to the importance of the use of microwaves in the developments which have taken place and which will continue to take place, a rather extensive treatment of microwave transmission by means of rectangular and cylindrical wave guides and coaxial cables is presented. This treatment has been included also because of the increasing importance of a knowledge of Maxwell's equations. It seems clear that, as more progress is made into the field of the hyper-frequencies, every electrical engineering student should have at least an elementary knowledge of the field theory to provide him with the necessary background for more advanced work. The treatment presented herewith is designed to lead the student into the elements of this theory and at the same time to provide practical applications in modern high-frequency transmission.

An effort has been made to cover the essentials of transmission and to lead logically to such subjects as filters, impedance matching, and wave guides. No attempt has been made, however, to present a complete treatment of any particular field. The books and periodicals to which references are made are suggested as additional reading material for those students who wish to progress further into the study of these subjects. The present work is written on the basis of two-dimensional fields in the development and use of equations, and at high frequencies a three-dimensional field arises which would alter certain concepts as presented in this text. However, to keep the work on an undergraduate level we choose to restrict it to a two-dimensional field basis. Also no account has been taken of the fact that a 2-wire line can become a

radiator of energy when the dimensions of the line are comparable to the wavelength of the transmitted frequency.

A knowledge of calculus and the elements of a-c theory on the part of the student has been assumed. More advanced mathematics is needed for certain portions of the text, and such material, as needed, has been included in suitable appendixes which may be either assigned or omitted according to the students' mathematical background. For those whose knowledge of electromagnetic theory is limited an appendix is devoted to the development, in a simple manner, of Maxwell's equations in the form in which they are most useful for the present treatment of wave guides and coaxial cables. The many examples which are introduced throughout the text and the chapter problems are designed to illustrate definite points of the text material as well as to provide problem exercises.

Since the text treats of communication circuits from the low voice frequencies through the microwave region it is believed that it will serve nicely as introductory material for any field of communication which the student proposes to enter.

In revising the text for the third edition, the authors have sought to increase the usefulness of the book further by the addition of considerable new material and by a change in certain treatments to conform with the procedures which are becoming standard now on account of the great advances made during World War II. Chapter I has been almost entirely rewritten and greatly improved. The chapter on impedance matching has again been extended and brought more nearly up to date. To conform with the content of more advanced courses the treatment of attenuation in wave guides has been based on Poynting's theorem. An important advance is the addition of many new problems, and an extension of the discussion in many places where experience of the past several years has indicated that such would be helpful.

The authors wish to express their appreciation to Professor G. F. Corcoran of the University of Maryland, and to Professor T. J. Higgins of the University of Wisconsin, for their many valuable suggestions and aid given during the preparation of this third edition.

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INTRODUCTION

The transfer of information constitutes the basis of modern civilization. The theory associated with the operation of transmission lines is fundamental to the general problem of transfer of intelligence and arises in all forms of electrical communication systems. Not only is the theory important in communication but it also has its application in the equally important transmission of power. Only in communication, however, due primarily to the wide range of frequencies involved, does it reach its greatest complexity and, it may be added, its greatest interest. It must be noted, however, that, of the two general types of electrical communication, (1) telephonic or wired and (2) radio or wireless, radio involves some of the most difficult of wire-transmission problems. Thus in any field where it is necessary to transmit information, the wire-transmission line is of basic importance and well merits the attention of every student of electrical engineering.

The transmission of information differs in one important aspect from the transmission of power, and this point is one which has often been passed over too lightly. Basically, from a power standpoint, the telephone line is one of very low efficiency while the power line should be of very high efficiency. In order to understand this distinction it is advisable to consider briefly the essentials of telephone transmission. The only way in which a signal may be transmitted to a distant point is by producing at the distant receiving end a change of some kind which will be correctly interpreted as meaning a definite event at the sending end. In a power line under constant load and stable conditions the only thing known at the receiving end about the sending end is that a generator is operating at a certain frequency and at approximately a certain voltage. If the receiving-end voltage suddenly drops to zero it may be inferred that something has thrown the generator off the line or that the line has broken down. In any case the information available at the receiving end of a power line concerning the sending end is distinctly limited.

In the telephone line, using the term in the general sense of meaning any communication line, the information relayed to the receiver must be supplied in the form of fluctuations of voltage or frequency, or both. In the simplest case this may be brought about by using a key or switch at the sending end which will make and break the circuit in a

previously agreed manner for the various letters of the alphabet. This is the method of the telegraph line. On the other hand, the frequency and amplitude of the generated wave may be varied in such a way as to produce the sound of the voice at the receiver end. The important point is that a telephone line should, in general, be capable of transmitting unchanged, or nearly so, a rather large number of frequencies of varying amplitudes. With the telegraph line above mentioned there is still this problem of transmitting a wide frequency band because the square-top pulse produced by operating the key is made up of a very wide band of frequencies. In general, however, this band is intentionally limited in width to strike a balance between quality and cost of transmission. It is thus seen that a faithful reproduction of the sending-end wave, even if it involves only a very small amount of power, is all that is required at the receiving end. Thus we may not be as concerned about the power received as we are about the fidelity of reproduction. This is in distinct contradiction to the power line where power is the important element. It must not be thought, however, that receiver-end power in communication lines is to be disregarded.

Communication transmission may be looked upon in two different ways. In a sense it is a problem in transients where it is required to find the shape of the pulse received when a given pulse is introduced at the sending end, or it may be, as is usually the case, a problem of transmission of a great number of frequencies each taken separately.[1] Part of the problem is to show that if transmission phenomena can be determined for a single frequency, arbitrarily selected somewhere in the frequency range, they are solvable for the band of frequencies. This is true for the ordinary line except for the effects of such phenomena as interference and interchannel modulation. Thus to begin with there exists a single frequency problem whose solution must be approached through a series of simple developments.

[1] See Appendix I.

CHAPTER I

TRANSMISSION-LINE PARAMETERS

Transmission-line problems are concerned with some combination of the four fundamental parameters: self-inductance L, wire-to-wire capacitance C, series resistance R, and shunt conductance G. The transmission of voice frequencies involves all four parameters since a transmission line is composed of the three interrelated circuits — the electric, the magnetic, and the dielectric. The electric circuit is formed by the wires of the line whereas the magnetic and dielectric circuits lie in the medium surrounding the wires. In transmission-line calculations the four parameters are assumed to be constants of the circuit; that is, they are independent of the current which may be flowing through the line. The present treatment will be limited to linear circuits only.

A transmission line, although appearing simple, is a somewhat complex circuit in which all four parameters, L, C, R, and G, are so interrelated as to lead to rather involved theoretical developments and to many difficulties in actual practice.

The present chapter is concerned with a brief review of some of the facts connected with L, C, R, and G of a line. It is assumed that the student has already dealt with this material elsewhere, so that brevity may seem to be the rule. The aim is to arrive with as little disturbance as possible at the fundamental T and π configurations of *sections* of a line, and it should be kept in mind that these fundamental T and π sections are in reality the starting points for the major parts of this text.

1. The Essentials of a Line. Two parallel wires of comparatively low resistance, properly supported and insulated from each other, make up the simple transmission line. On such a line it is only necessary to connect a suitable transmitter to one end and a receiver to the other. If the line is not too long, a change in the applied voltage at the sending end will be noted by an appropriate change in the reading of an ammeter at the receiving end. If a variable-frequency generator is connected to the sending end, a suitable piece of equipment connected at the receiving end will make a record of any change in either the frequency or the magnitude of the generated emf. Thus for a short line, little difficulty is encountered in the transmission of the requisite elements of communication.

As the line is lengthened certain difficulties begin to arise. In the first place, if the line is very long and if any reference to circuit parameters other than resistance is omitted for the present, it is found that the resistance alone soon becomes so great that a very high voltage would be required at the sending end in order to obtain a readable current at the receiving end. As an illustration, suppose that two lines are made of No. 19 AWG copper wire and the lengths of the lines are 50 feet and 1000 miles. Assume identical generators having an internal resistance of 50 ohms and a generated emf of 10 volts to be connected to each line. (In order to obtain maximum current readings, a milliammeter of zero internal resistance will be assumed to be connected directly across the receiving end of each line.) The resistance of the 50-foot line is 0.805 ohm, and the received current is 197 milliamperes, a value which is easily measured. The resistance of the 1000-mile line is 85,020 ohms, and the received current is 0.117 milliampere, a value which is below the range of the ordinary indicating instrument. The long line, therefore, is seen to require some kind of auxiliary equipment in order to make successful communication possible.

Actually, as the line is lengthened, reactance due to self-inductance and reactance due to capacitance become serious. Also a certain amount of leakage of current from one wire to the other begins to be objectionable. Since the reactances are functions of the frequency transmitted, it is immediately seen that a new kind of difficulty has come into being because the transmission characteristics of the line, being functions of the reactances, become different for different frequencies. In the transmission of ordinary speech frequencies, for instance, part of the frequency range will be treated differently from other parts, that is, will arrive sooner or later, or will be of lower or higher relative intensity. It thus becomes necessary to investigate the fundamental parameters of a line in order to determine just how important they are in transmission-line calculations.

2. Self-Inductance of Conductors in Terms of $N\phi/i$. The self-inductance of a conductor may be evaluated in terms of the self flux linkages ($N\phi$) established by the conductor per unit current flow in the conductor. A *unit* flux linkage is one unit of magnetic flux linking with or encompassing the *entire conductor current*. Thus ϕ_{ext} shown in Fig. 1–1 represents a unit flux linkage with the conductor if $\phi_{ext} = B_x A_x =$ unity, because the flux in question links the entire conductor current. In a quantitative sense, ϕ_{ext} in the figure is distributed over the entire area $(dx \cdot 1)$.

The flux ϕ_{int} shown in the figure encompasses only a fractional part of the entire conductor current and as such represents only a *partial* flux

linkage with the conductor. *If the current density is uniform over the* cross-sectional area of the conductor, ϕ_{int} encompasses or links with $\pi x^2/\pi r^2$ fractional part of the conductor current, and hence N, in the expression $L = N\phi/i$, is reduced by the ratio of $\pi x^2/\pi r^2$ to 1. The value of x in the above fraction is of course less than or equal to r since internal flux linkages are being considered.

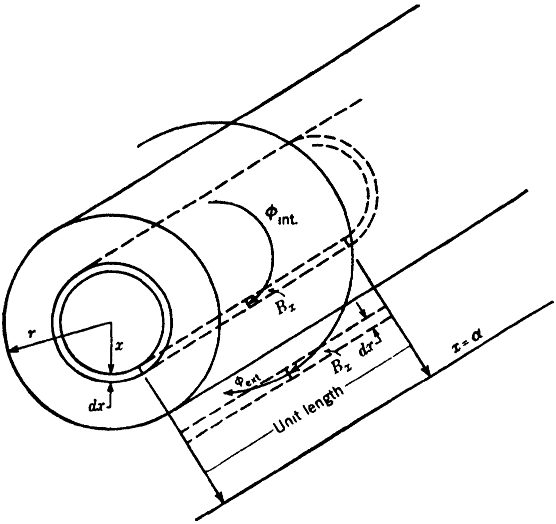


FIG. 1–1. Elemental length of conductor showing ϕ_{int} and ϕ_{ext}.

The concept of *partial* flux linkage is important in the calculation of inductance of all conductors since the internal flux linkages of a conductor are usually an appreciable percentage of the total flux linkages. This percentage is particularly high in coaxial cables at the lower frequencies.

The inductance of the conductor shown in Fig. 1–1 due to the internal flux linkages is usually calculated on the basis of unit length of conductor and is

$$L_{\text{int}} = \sum \frac{N\phi}{i} = \int_0^r \frac{\dfrac{\pi x^2}{\pi r^2} B_x dx}{I} \text{ per unit length} \qquad [1\text{–}1]$$

where I is the total conductor current,
$B_x = \mu_0 \mu_r H_x$ (magnetic flux density),
μ_0 = permeability of free space ($4\pi \times 10^{-7}$ in rationalized mks units),
μ_r = relative permeability of the conductor material.

Since internal flux linkages are being considered where $x < r$,

$$H_x = \frac{I_x}{2\pi x} \quad \text{(magnetic field intensity[1])}$$

$$I_x = \frac{\pi x^2}{\pi r^2} \cdot I \quad \text{(the current linked by } H_x\text{)}$$

Thus B_x in equation 1–1 becomes

$$\mu_0 \mu_r \frac{Ix}{2\pi r^2} \text{ webers/sq m} \quad \text{(in mks units)}$$

and

$$L_{\text{int}} = \mu_0 \mu_r \int_0^r \frac{\pi x^2}{\pi r^2} \cdot \frac{x}{2\pi r^2} \cdot dx = \frac{\mu_0 \mu_r}{8\pi} \qquad [1\text{–}2]$$

$$= \frac{\mu_{r1} \times 10^{-7}}{2} \text{ henry/meter} \qquad [1\text{–}3]$$

where μ_{r1} is plainly the relative permeability of the conductor material.

The self-inductance of the conductor per unit length due to external flux linkages from $x = r$ to $x = \alpha$ is

$$L_{\text{ext}} = \sum \frac{N\phi}{i} = \int_r^\alpha \frac{(1) \times (B_x dx)}{I} \text{ per unit length} \qquad [1\text{–}4]$$

where

$$B_x = \mu_0 \mu_r H_x = \mu_0 \mu_r \frac{I}{2\pi x} \quad (x > r)$$

Hence L_{ext} (from $x = r$ to $x = \alpha$) is

$$L_{\text{ext}} = \frac{\mu_0 \mu_r}{2\pi} \int_r^\alpha \frac{dx}{x} = \frac{\mu_0 \mu_r}{2\pi} \ln \frac{\alpha}{r} \qquad [1\text{–}5]$$

$$= 2\mu_{r2} \times 10^{-7} \ln \frac{\alpha}{r} \text{ henry/meter} \qquad [1\text{–}6][2]$$

where μ_{r2} is the relative permeability of the medium outside the conductor.

[1] In general, rationalized mks units are employed. In this system of units H is expressed directly in ampere-turns per meter and B in webers per square meter. $H_x = I_x/2\pi x$ follows directly from the circuital law of magnetism; that is, the line integral of the magnetic field intensity taken around any closed path equals the current enclosed.

[2] Throughout this text the symbol ln will mean logarithm to the base ϵ, and log will mean logarithm to the base 10.

The total self-inductance of the conductor per meter length due to internal and external flux linkages out to $x = \alpha$ is $L_{int} + L_{ext}$ or

$$L_{self} = \left(\frac{\mu_{r1}}{2} + 2\mu_{r2} \ln \frac{\alpha}{r}\right) \times 10^{-7} \text{ henry/meter} \qquad [1\text{-}7]$$

3. Self-Inductance of Parallel Wires. Communication lines, the type primarily under consideration in this text, are usually composed of two parallel wires, and as such it is necessary to derive an expression giving the total self-inductance of the pair. The flux set up by the current I_1 of the two-conductor line shown in Fig. 1-2 may be divided into four components[3]: (1) internal flux within the conductor itself, (2) flux passing between the two wires, (3) flux crossing the second wire, and (4) flux surrounding both wires. Similarly, the second conductor also sets up a flux which may be divided into corresponding component parts. The flux which surrounds both conductors does not link any net current and hence produces no flux linkages. Some of the flux which crosses the second conductor surrounds nearly all the current I_2 of the second conductor while some surrounds only a very small portion. The flux which passes between the wires links the total current I_1, and in determining the external flux linkages the integration is carried from $x = r$ to $x = d$, the center of the second conductor, thus taking into effect, approximately, the flux linkages which have part of their paths through the second conductor. In more advanced texts it is proved that for nonmagnetic conductors this integration gives exact results.

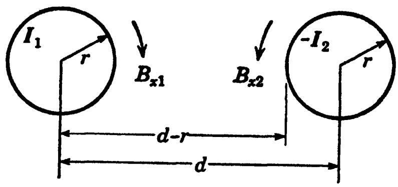


FIG. 1-2. Parallel conductors having separation d. (I_1 equals I_2 in magnitude.)

The self-inductance of a parallel-wire line, taking into consideration both conductors then becomes

$$L = 2\left(\frac{\mu_{r1}}{2} + 2\mu_{r2} \ln \frac{d}{r}\right) \times 10^{-7} \text{ henry/loop meter} \qquad [1\text{-}8]$$

where α of equation 1-5 is now equal to d. If the conductors are of magnetic material a small error exists in this equation which becomes negligible when $d \gg r$.

[3] In general electromagnetic theory the basic definition of the inductance of a circuit is defined as that parameter which if multiplied by $i^2/2$ yields the energy W stored in the magnetic field of the circuit when carrying current i. Thus $W = Li^2/2$, or $L = 2W/i^2$, and the calculation of L depends on the calculation of W rather than on flux linkages per unit current. Since, however, the flux-linkage concept is the one normally employed in elementary courses, it will be used here.

In terms of logarithm to the base 10,

$$L = \left(\mu_{r1} + 9.210\mu_{r2} \log \frac{d}{r}\right) \times 10^{-7} \text{ henry/loop meter} \qquad [1\text{-}9]$$

If nonferromagnetic materials are involved, as is usually the case,

$$L = \left(1 + 9.210 \log \frac{d}{r}\right) \times 10^{-7} \text{ henry/loop meter} \qquad [1\text{-}10]$$

or

$$L = \left(0.1609 + 1.482 \log \frac{d}{r}\right) \times 10^{-3} \text{ henry/mile of line} \qquad [1\text{-}11]$$

As the frequency and radii of the conductors increase the internal self-inductance decreases in accordance with a relationship which is derived in Art. 11 and illustrated in graphical form in Fig. 1–9. At high frequencies where skin effect is very pronounced the internal self-inductance term of equations 1–8 through 1–11 is negligible, and

$$L = 4 \ln \frac{d}{r} \times 10^{-7} \text{ henry/loop meter} \qquad [1\text{-}12]$$

or

$$L = 1.482 \log \frac{d}{r} \times 10^{-3} \text{ henry/mile} \qquad [1\text{-}13]$$

for nonferromagnetic materials.

As an illustrative example, calculate the inductance of a transmission line made of No. 19 AWG copper wires spaced 12 inches center to center. This wire has a radius of $r = 0.01794$ inch. The use of equation 1–11 leads to the following:

$$\frac{d}{r} = \frac{12.0}{0.01794} = 669$$

and

$$\begin{aligned} L &= (0.1609 + 1.482 \log 669) \times 10^{-3} \\ &= (0.1609 + 4.19) \times 10^{-3} \\ &= 0.00435 \text{ henry/mile of line} \end{aligned}$$

4. Self-Inductance of Coaxial Cables. In the coaxial cable one conductor is a hollow cylinder whereas the other is a cylindrical rod inside the outer cylinder and concentric with it. These two conductors,

being parts of the same circuit, carry the same current; thus, if the current in one is I, the current in the other is $-I$.

The self-inductance of the inner conductor L_{in} is due to three distinct types of linkages. (See Fig. 1-3.)

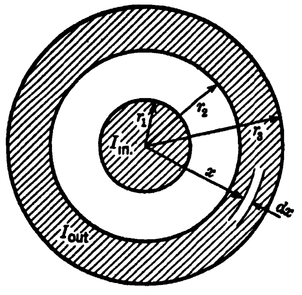


FIG. 1-3. Coaxial pair.

(*a*) In the region from $r = 0$ to $r = r_1$, the internal flux leads to an expression for self-inductance similar to equation 1-2,

$$L\Big]_0^{r_1} = \mu_0\mu_{r1}\int_0^{r_1} \frac{\pi x^2}{\pi r_1^2}\cdot\frac{x}{2\pi r_1^2}\cdot dx = \frac{\mu_{r1} \times 10^{-7}}{2} \text{ henry/meter} \quad [1\text{-}14]$$

(*b*) In the region from r_1 to r_2, from equation 1-5,

$$L\Big]_{r_1}^{r_2} = \frac{\mu_0\mu_{r2}}{2\pi}\int_{r_1}^{r_2}\frac{dx}{x} = 2\mu_{r2} \times 10^{-7} \times \ln\frac{r_2}{r_1} \text{ henry/meter} \quad [1\text{-}15]$$

where α has been replaced by the radius r_2.

(*c*) In considering a linkage which lies between r_2 and r_3 it is noted that the current linked is all of the current in the inner conductor I_{in} plus a fractional part of the current in the outer conductor. Again, on the basis that the current is uniformly distributed over the cross section of the conductor, the current linked is

$$I_{in} + \left[\frac{\pi x^2 - \pi r_2^2}{\pi r_3^2 - \pi r_2^2}\right] I_{out} = I_{in}\left[1 - \frac{x^2 - r_2^2}{r_3^2 - r_2^2}\right] = I_{in}\left[\frac{r_3^2 - x^2}{r_3^2 - r_2^2}\right]$$

where $I_{out} = -I_{in}$. Accordingly the self-inductance for this region is

$$L\Big]_{r_2}^{r_3} = \frac{\mu_0\mu_{r3}}{2\pi}\int_{r_2}^{r_3}\frac{r_3^2 - x^2}{r_3^2 - r_2^2}\cdot\frac{dx}{x}$$

$$= \frac{\mu_0\mu_{r3}}{2\pi(r_3^2 - r_2^2)}\left[r_3^2 \ln\frac{r_3}{r_2} - \frac{1}{2}(r_3^2 - r_2^2)\right]$$

$$= \frac{2\mu_{r3}}{r_3^2 - r_2^2}\left[r_3^2 \ln\frac{r_3}{r_2} - \frac{1}{2}(r_3^2 - r_2^2)\right] \times 10^{-7} \text{ henry/meter} \quad [1\text{-}16]$$

The total self-inductance for the inner conductor thus becomes

$$L_{\text{in}} = \left\{\frac{\mu_{r1}}{2} + 2\mu_{r2} \ln \frac{r_2}{r_1} + \frac{2\mu_{r3}}{r_3^2 - r_2^2}\left[r_3^2 \ln \frac{r_3}{r_2} - \frac{1}{2}(r_3^2 - r_2^2)\right]\right\} \times 10^{-7}$$

henry/meter [1–17]

The inductance due to the current in the outer conductor may be determined in a somewhat similar manner. In this derivation there will be only one term, that due to flux in the region between r_2 and r_3, because a uniform current flowing in a hollow conductor cannot produce flux inside the cylinder.[4] The fractional part of a complete linkage of the flux at $r = x$ with the current I_{out} in the outer conductor is

$$\frac{\pi x^2 - \pi r_2^2}{\pi r_3^2 - \pi r_2^2} = \frac{x^2 - r_2^2}{r_3^2 - r_2^2}$$

The current producing the flux through the element dx is

$$\left[\frac{\pi x^2 - \pi r_2^2}{\pi r_3^2 - \pi r_2^2} I_{\text{out}} + I_{\text{in}}\right] = \left[\frac{\pi x^2 - \pi r_2^2}{\pi r_3^2 - \pi r_2^2} - 1\right] \cdot I_{\text{out}} = \left[\frac{\pi x^2 - \pi r_3^2}{\pi r_3^2 - \pi r_2^2}\right] \cdot I_{\text{out}}$$

where $I_{\text{in}} = -I_{\text{out}}$. Thus the self-inductance of the outer conductor becomes

$$L_{\text{out}} = \frac{\mu_0 \mu_{r3}}{2\pi} \int_{r_2}^{r_3} \frac{x^2 - r_2^2}{r_3^2 - r_2^2} \cdot \frac{x^2 - r_3^2}{r_3^2 - r_2^2} \cdot \frac{dx}{x}$$

$$= \frac{2\mu_{r3}}{r_3^2 - r_2^2}\left[\frac{r_2^2 r_3^2}{r_3^2 - r_2^2} \ln \frac{r_3}{r_2} - \frac{1}{4}(r_3^2 + r_2^2)\right] \times 10^{-7}$$

henry/meter [1–18]

The total self-inductance of the coaxial pair is the sum of the expressions 1–17 and 1–18.

$$L = \left\{\frac{\mu_{r1}}{2} + 2\mu_{r2} \ln \frac{r_2}{r_1} + \frac{2\mu_{r3}}{r_3^2 - r_2^2}\left[\frac{r_3^4}{r_3^2 - r_2^2} \ln \frac{r_3}{r_2} + \frac{1}{4} r_2^2 - \frac{3}{4} r_3^2\right]\right\} \times 10^{-7}$$

henry/meter [1–19]

Equation 1–19 yields the correct total self-inductance of a coaxial line at low frequencies (uniform current densities in the conductor), but for practical values of r_2 and r_3 the third term is negligibly small, and so the

[4] This fact follows directly from the circuital law of magnetism since the line integral of free-space magnetic intensity around any closed path in a magnetic field is equal to zero, provided the path of integration does not link with a current-carrying region. The path of integration around the inside of the conductor would not link a current-carrying region; hence H is zero inside the conductor.

low-frequency self-inductance is expressed as

$$L \doteq \left(\frac{\mu_{r1}}{2} + 2\mu_{r2} \ln \frac{r_2}{r_1}\right) \times 10^{-7} \text{ henry/meter} \qquad [1\text{–}20]$$

If nonferromagnetic materials are employed,

$$L = \left(\frac{1}{2} + 2 \ln \frac{r_2}{r_1}\right) \times 10^{-7} \text{ henry/meter} \qquad [1\text{–}21]$$

or

$$L = \left(0.08045 + 0.741 \log \frac{r_2}{r_1}\right) \times 10^{-3} \text{ henry/mile} \qquad [1\text{–}22]$$

For very high frequencies and thus for high skin effect, the current is confined essentially to the inner surface of the outer conductor and to the outer surface of the inner conductor. This leads to a condition where there is practically no flux inside the inner conductor and where the effective thickness of the outer conductor is practically zero. Thus the terms in equations 1–17 and 1–18 reduce to the single term:

$$L = 2\mu_{r2} \ln \frac{r_2}{r_1} \times 10^{-7} \text{ henry/meter} \qquad [1\text{–}23]$$

For nonferromagnetic material and using a for the radius of the inner conductor and b for the inner radius of the outer conductor,

$$L = 2 \ln \frac{b}{a} \times 10^{-7} \text{ henry/meter} \qquad [1\text{–}24]$$

or

$$L = 0.741 \log \frac{b}{a} \times 10^{-3} \text{ henry/mile} \qquad [1\text{–}25]$$

Note that the inductance as given by equation 1–25 is approximately one-half that for two parallel wires separated by a distance b as given by equation 1–11.

Illustrative Example. The low-frequency self-inductance of a coaxial cable will be calculated giving the components of L_{in} and L_{out}, thereby showing that equation 1–20 is valid for the particular cable being considered. For the coaxial cable $r_1 = 1/64$, $r_2 = 1/8$, and $r_3 = 5/32$, all in inches.

From equation 1–17, for nonmagnetic materials

$$L_{\text{in}} = \left\{\frac{1}{2} + 2 \ln \frac{r_2}{r_1} + \frac{2}{r_3^2 - r_2^2}\left[r_3^2 \ln \frac{r_3}{r_2} - \frac{1}{2}(r_3^2 - r_2^2)\right]\right\} \times 10^{-7}$$

$$= (0.5 + 2 \ln 8 + 0.239) \times 10^{-7} = (0.5 + 4.16 + 0.239) \times 10^{-7}$$

$$= 4.899 \times 10^{-7} \text{ henry/meter}$$

From equation 1–18, for nonmagnetic materials,

$$L_{\text{out}} = \frac{2}{r_3^2 - r_2^2}\left[\frac{r_2^2 r_3^2}{r_3^2 - r_2^2}\ln\frac{r_3}{r_2} - \frac{1}{4}(r_3^2 + r_2^2)\right] \times 10^{-7}$$

$$= -0.074 \times 10^{-7} \text{ henry/meter}$$

$$L = L_{\text{in}} + L_{\text{out}} = 4.825 \times 10^{-7} \text{ henry/meter}$$

From equation 1–20, for nonmagnetic materials,

$$L \doteq \left(\frac{1}{2} + 2\ln\frac{r_2}{r_1}\right) \times 10^{-7}$$

$$\doteq (0.5 + 2\ln 8) \times 10^{-7} = 4.66 \times 10^{-7} \text{ henry/meter}$$

It is to be observed that the net effect of the third term of equation 1–19 is to add 0.165×10^{-7} to 4.66×10^{-7} henry per meter or an increase of 3.5 per cent in the low-frequency value.

The high-frequency value of self-inductance is given by equation 1–23 and for nonmagnetic materials is

$$L = 2\ln\frac{r_2}{r_1} \times 10^{-7}$$

$$= 4.16 \times 10^{-7} \text{ henry/meter}$$

which is 86.4 per cent of the true low-frequency value.

5. Capacitance of Parallel-Wire and Coaxial Lines. *Parallel-Wire Lines.* The capacitance of two parallel wires which are separated by more than 10 times the radius of the wires may be determined directly from the basic definition of capacitance, namely:

$$C = \frac{q}{V}$$

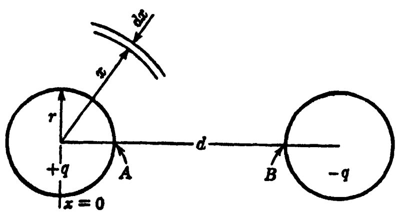


Fig. 1–4. Parallel conductors having separation d.

In Fig. 1–4 it is only necessary to assign $+q$ units of charge per unit length to one conductor and $-q$ units of charge per unit length to the other. A cylindrical Gaussian surface of radius x and unit length enclosing the $+q$ conductor will then be pierced by $+q$ coulombs of electric flux, the density of which is

$$D_{+q} = \frac{q}{2\pi x} \text{ coulombs/sq m} \quad (\text{where } x > r) \qquad [1\text{–}26]$$

The corresponding electric intensity vector at any distance x from the

center of the $+q$ conductor is

$$\mathcal{E}_{+q} = \frac{q}{\epsilon_0 \epsilon_r 2\pi x} \text{ volts/meter} \tag{1-27}$$

where

ϵ_0 is the permittivity of free space $\left(\frac{1}{36\pi \times 10^9} \text{ in rationalized mks units}\right)$,

ϵ_r is the relative permittivity of the medium between the two conductors.

An expression similar to equation 1–27 may be written for $\mathcal{E}_{-q}$ and the voltage drop from A to B in Fig. 1–4 evaluated as

$$V = \frac{q}{\epsilon_0 \epsilon_r 2\pi}\left[\int_r^{d-r} \frac{dx}{x} - \int_{d-r}^{r} \frac{dy}{y}\right]$$

$$= \frac{q}{\epsilon_0 \epsilon_r \pi} \ln \frac{d-r}{r} \text{ volts} \tag{1-28}$$

where the first integral accounts for the effect of the charge on wire A and the second for the charge on B. Since the path of integration may be along any path connecting the two wires, the line connecting the centers of the wires is chosen since it is the simplest path.

The approximate wire-to-wire capacitance per unit length of line is

$$C_{w\text{-to-}w} = \frac{q}{V} = \frac{\epsilon_0 \epsilon_r \pi}{\ln \dfrac{d-r}{r}} = \frac{\epsilon_0 \epsilon_r \pi}{\ln \dfrac{d}{r}} \quad \text{when} \quad d \gg r \tag{1-29}$$

(See Art. 6 where proximity effect is taken into consideration.) Or, in more convenient forms,

$$C_{w\text{-to-}w} = \frac{\epsilon_r}{36 \times 10^9 \ln \dfrac{d-r}{r}} \doteq \frac{\epsilon_r}{36 \times 10^9 \ln \dfrac{d}{r}} \text{ farad/meter} \tag{1-30}$$

$$C_{w\text{-to-}w} = \frac{27.78\epsilon_r}{\ln \dfrac{d-r}{r}} \doteq \frac{27.78\epsilon_r}{\ln \dfrac{d}{r}} \; \mu\mu\text{f/meter} \tag{1-31}$$

$$C_{w\text{-to-}w} = \frac{0.0194\epsilon_r}{\log \dfrac{d-r}{r}} \doteq \frac{0.0194\epsilon_r}{\log \dfrac{d}{r}} \; \mu\text{f/mile} \tag{1-32}$$

when $d \gg r$.

The above formulas for wire-to-wire capacitance have been derived by considering that the conductors are far removed from the ground plane.

If the wires are strung close to the ground plane, the presence of the ground increases the wire-to-wire capacitance to some extent. For example, two No. 0000 AWG wires spaced 8 feet, center to center, have a wire-to-wire capacitance of 0.0074 microfarad per mile, neglecting the presence of ground. If these wires are located only 10 feet above the ground plane, the wire-to-wire capacitance is increased by 1.3 per cent or to 0.0076 microfarad per mile. The per cent increase in the wire-to-wire capacitance of the ordinary communication line due to the presence of ground is usually negligibly small and is not considered further in this text. (For details, see *Basic Electrical Engineering* by G. F. Corcoran, p. 364, John Wiley & Sons, New York, 1949.)

Illustrative Example. It is desired to determine the capacitance, wire to wire, in microfarads per mile, of a line constructed of No. 19 AWG wires spaced 12 inches center to center in air. For this wire and spacing,

$$\frac{d}{r} = \frac{12.0}{0.01794} = 669$$

and

$$\log \frac{d}{r} = 2.825$$

Equation 1–32 becomes, since $\epsilon_r = 1$

$$C_{w\text{-to-}w} = \frac{0.0194}{2.825} = 0.00687 \ \mu f/\text{mile}$$

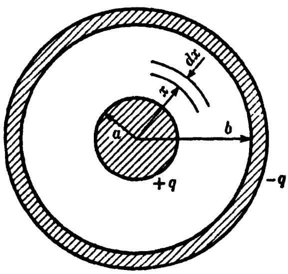


Fig. 1–5. Coaxial pair.

Coaxial Lines. The capacitance of the two conductors of a coaxial line or pair, Fig. 1–5, may be obtained directly from the derivation given above for parallel-wire lines if it is observed that the charge on the outer conductor of the coaxial line establishes no electric field in the region between the conductors. (This follows directly from Gauss' theorem.)

The expression for the electric-field intensity in the region between the two conductors is given by equation 1–27, and the voltage difference between the conductors due to the assigned + and − charges is

$$V = \frac{q}{\epsilon_0 \epsilon_r 2\pi} \int_a^b \frac{dx}{x} = \frac{q}{\epsilon_0 \epsilon_r 2\pi} \ln \frac{b}{a} \qquad [1\text{–}33]$$

The capacitance of the two conductors per unit length of line is

$$C_{co} = \frac{q}{V} = \frac{\epsilon_0 \epsilon_r 2\pi}{\ln \dfrac{b}{a}} \tag{1-34}$$

or, in more convenient forms,

$$C_{co} = \frac{\epsilon_r}{18 \times 10^9 \ln \dfrac{b}{a}} \text{ farad/meter} \tag{1-35}$$

$$C_{co} = \frac{55.56\epsilon_r}{\ln \dfrac{b}{a}} \ \mu\mu\text{f/meter} \tag{1-36}$$

$$C_{co} = \frac{0.0388\epsilon_r}{\log \dfrac{b}{a}} \ \mu\text{f/mile} \tag{1-37}$$

Illustrative Example. Let it be required to determine the capacitance of the coaxial cable given in the illustrative example of Art. 4 in which ϵ_r is unity, the dielectric being air. From equation 1–36,

$$C_{co} = \frac{55.56}{\ln 8} = 26.7 \ \mu\mu\text{f/meter}$$

where

$$a = r_1 = \tfrac{1}{64} \text{ in.}$$

$$b = r_2 = \tfrac{1}{8} \text{ in.}$$

or, from equation 1–37,

$$C_{co} = \frac{0.0388}{\log 8} = 0.043 \ \mu\text{f/mile}$$

6. Parallel-Wire Capacitance Taking Proximity Effect into Consideration. If the center-to-center separation of the cylindrical conductors shown in cross section in Fig. 1–6 is less than about 10 times the radius of the conductors, the attraction of the $+q$ and $-q$ charges effects a sensible redistribution of the line charges which in turn affects the wire-to-wire capacitance to some extent. The redistribution of charge is indicated in Fig. 1–6 by $s < d$ where s is the separation of the line charges which are employed to represent the actual charge distributions present on the cylindrical conductors. (In the derivation of equation 1–29, it was tacitly assumed that the actual surface charges could be

replaced with line charges which coincided with the centers of the conductors.)

In Fig. 1–6, the potential difference between surface 1 and surface 2 due to both the $+q$ and $-q$ line charges, is

$$V_{12} = 2\left(\frac{q}{2\pi\epsilon_0\epsilon_r}\ln\frac{b'}{a'}\right) = \frac{36 \times 10^9 q}{\epsilon_r}\ln\frac{b'}{a'} \text{ volts} \qquad [1\text{–}38]$$

The potential difference V_{12} is of course a constant and could be evaluated directly, provided b' and a' were known. Since b' and a' are unknown at this stage of the derivation, it is desirable to show that with the line charges separated by a distance s there exist equipotential

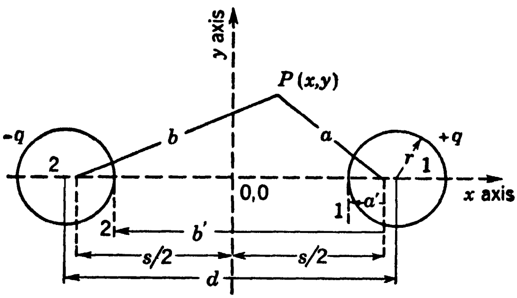


Fig. 1–6. Redistribution of charge for calculation of proximity effect.

surfaces which are circular in cross section, one pair of which has centers located at $x = d/2$ and $x = -d/2$. And these two equipotential surfaces are of exactly the correct size to accommodate the two line conductors which are themselves equipotential surfaces. To this end, the potential at *any* point in the vicinity of the two line charges, such as point P in Fig. 1–6, is written as

$$V_P = \frac{18 \times 10^9 q}{\epsilon_r}\ln\frac{b}{a} \text{ volts (due to both } +q \text{ and } -q) \qquad [1\text{–}39]$$

For V_P to be constant,

$$\frac{b}{a} = k \qquad [1\text{–}40]$$

where k is a constant to be determined.

It follows from Fig. 1–6 that

$$b = \sqrt{\left(\frac{s}{2} + x\right)^2 + y^2} \quad \text{and} \quad a = \sqrt{\left(\frac{s}{2} - x\right)^2 + y^2}$$

From equation 1–40,

$$\left(\frac{s}{2} + x\right)^2 + y^2 = k^2\left[\left(\frac{s}{2} - x\right)^2 + y^2\right] \qquad [1\text{–}41]$$

It follows that

$$x^2 - \frac{(k^2 + 1)}{(k^2 - 1)}\cdot sx + \frac{(k^2 + 1)^2}{(k^2 - 1)^2}\cdot\frac{s^2}{4} + y^2 = -\frac{s^2}{4} + \frac{(k^2 + 1)^2}{(k^2 - 1)^2}\cdot\frac{s^2}{4} \qquad [1\text{–}42]$$

where the $\frac{(k^2 + 1)^2}{(k^2 - 1)^2}\cdot\frac{s^2}{4}$ term has been added to both sides of the equation in order to complete the square in x. Equation 1–42 may be written as

$$\left[x - \frac{(k^2 + 1)}{(k^2 - 1)}\cdot\frac{s}{2}\right]^2 + y^2 = \left[\frac{ks}{(k^2 - 1)}\right]^2 \qquad [1\text{–}43]$$

thus showing that the cross section of the equipotential surfaces established by the $+q$ and $-q$ line charges are circles. The conductor equipotential surfaces have radii equal to r; hence

$$r = \frac{ks}{k^2 - 1} \qquad [1\text{–}44]$$

Since the center of the $+q$ conductor is known to be at $y = 0$ and $x = d/2$, it follows from equation 1–43 that

$$\frac{d}{2} = \frac{k^2 + 1}{k^2 - 1}\cdot\frac{s}{2} \qquad [1\text{–}45]$$

Solving equations 1–44 and 1–45 for k in terms of the known values of d and r, there is obtained

$$k = \frac{b'}{a'} = \frac{d}{2r} \pm \sqrt{\frac{d^2}{4r^2} - 1} \qquad [1\text{–}46][5]$$

and

$$V_{12} = \frac{36 \times 10^9 q}{\epsilon_r}\ln\left(\frac{d}{2r} + \sqrt{\frac{d^2}{4r^2} - 1}\right) \qquad [1\text{–}47]$$

$$C = \frac{q}{V_{12}} = \frac{\epsilon_r}{36 \times 10^9 \ln\left(\frac{d}{2r} + \sqrt{\left(\frac{d}{2r}\right)^2 - 1}\right)}\ \text{farad/meter} \qquad [1\text{–}48]$$

[5] The use of the minus sign before the radical leads to a mathematical incongruity. When d is large compared to r, equation 1–48 reduces immediately to equation 1–30, but such is not the case if the minus sign is used before the radical.

Equation 1–48 yields results which are somewhat more accurate than the results obtained by equation 1–30, but where d is large compared to r the difference is negligible.

Equation 1–48 may be expressed in terms of the inverse hyperbolic cosine[6] of $d/2r$. Since tables of hyperbolic functions are readily available (see Appendix IX), the following form of equation 1–48 may be used to advantage:

$$C = \frac{\epsilon_r}{36 \times 10^9 \cosh^{-1} \dfrac{d}{2r}} \text{ farad/meter} \qquad [1\text{–}48a]$$

Illustrative Example. In Fig. 1–6 it will be assumed that the conductors are surrounded by air ($\epsilon_r = 1$) and that $d = 4$ cm and $r = 1$ cm. Let it be required to find the capacitance of the two conductors per meter length by means of equations 1–31, 1–48, and 1–48a, recognizing that equation 1–31 can yield only an approximate result since, in this case, the proximity effect cannot be neglected because of the low ratio of d to r.

The approximate result obtained by means of equation 1–31 is

$$C_{w-\text{to}-w} = \frac{27.78}{\ln \dfrac{d-r}{r}} = \frac{27.78}{\ln 3} = \frac{27.78}{1.1} = 25.2\ \mu\mu\text{f/meter}$$

From equation 1–48,

$$C_{w-\text{to}-w} = \frac{10^{12}}{36 \times 10^9 \ln \left[\dfrac{d}{2r} + \sqrt{\left(\dfrac{d}{2r}\right)^2 - 1}\right]} = \frac{10^3}{36 \ln [2 + \sqrt{2^2 - 1}]}$$

$$= \frac{27.78}{\ln 3.73} = \frac{27.78}{1.317} = 21.1\ \mu\mu\text{f/meter}$$

[6] The hyperbolic cosine of x (abbreviated *cosh* x) is by definition $\dfrac{\epsilon^x + \epsilon^{-x}}{2}$. (See Appendix III and Appendix IX.)

If $$\cosh x = \frac{\epsilon^x + \epsilon^{-x}}{2} = \frac{d}{2r} \quad \text{then} \quad \epsilon^{2x} - \frac{d}{r}\epsilon^x + 1 = 0$$

and $$\epsilon^x = \frac{d}{2r} \pm \sqrt{\left(\frac{d}{2r}\right)^2 - 1}$$

$$x = \cosh^{-1} \frac{d}{2r} = \ln\left(\frac{d}{2r} + \sqrt{\left(\frac{d}{2r}\right)^2 - 1}\right)$$

From equation 1–48a,

$$C_{w-to-w} = \frac{10^{12}}{36 \times 10^{9} \cosh^{-1} \dfrac{d}{2r}}$$

$$= \frac{10^{3}}{36 \times \cosh^{-1} 2} = \frac{27.78}{1.317} = 21.1 \ \mu\mu\text{f/meter}$$

where $\cosh^{-1} 2$ is obtained from the tabular values given on page 389.

7. Series Resistance. It is in the series resistance of a current-carrying line that the greatest percentage of the line loss occurs. The loss in question is the result of the irreversible transformation of electric energy to heat energy in the series resistance. If this loss is to be properly accounted for, the series resistance must be evaluated as accurately as possible for the particular conditions under which the line is operating.

The series resistance is a function of both the operating temperature and the operating frequency as well as the size of the line conductors and the conductor material. It will be remembered that in general for direct currents

$$R = \rho \frac{l}{A} \qquad [1\text{–}49]$$

where ρ is the resistivity of the conductor material,
l is the length of the conductor,
A is the cross-sectional area of the conductor.

As applied to a loop of a two-conductor line of length l,

$$R_{\text{loop}} = \rho_1 \frac{l}{A_1} + \rho_2 \frac{l}{A_2} \qquad [1\text{–}50]$$

where the subscripts 1 refer to one of the two conductors and the subscripts 2 to the other.

In practice the unit length of l is often selected as 1 foot and the unit of A is selected as 1 circular mil. In this case ρ is expressed in ohm-circular-mils per foot, which for standard annealed copper is 10.37 ohms at 20°C. For hard-drawn copper (97.3 per cent conductivity), ρ is 10.66 ohm-circular-mils per foot at 20°C. (See Appendix VIII for table of standard annealed copper and hard-drawn copper conductors.)

The value of ρ and hence of R varies considerably with temperature. For the case of copper the variation in resistance with variation of tem-

perature can be accounted for quite accurately by means of the following relationship:

$$\frac{R_2}{R_1} = \frac{240 + t_2}{240 + t_1} \qquad [1\text{–}51]$$

where R_1 and R_2 are the resistances at the temperatures t_1 and t_2 degrees centigrade, respectively. The number 240 in equation 1–51 is actually the reciprocal of the zero-degree-centigrade value of the temperature coefficient of resistivity. The coefficient is 0.00427 for annealed copper and 0.00415 for hard-drawn copper; hence 240 represents a compromise between these two grades of copper.

In derivations which are carried out in mks units it is sometimes desirable to know the resistivity of the material in ohm-square-meters per meter. If the resistivity in ohm-circular-mils per foot (ρ_{cmf}) is known, it follows that the resistivity in ohm-square-meters per meter (ρ_m) is

$$\begin{aligned} \rho_m &= \rho_{cmf}\left(\frac{\text{sq m}}{\text{cir mils}}\right)\left(\frac{\text{ft}}{\text{meter}}\right) \\ &= \rho_{cmf}(5.06 \times 10^{-10})(3.28) \\ &= 16.6 \times 10^{-10}\rho_{cmf} \end{aligned}$$

For the case of hard-drawn copper the resistivity is

$$\begin{aligned} \rho_m &= 16.6 \times 10^{-10} \times 10.66 \\ &= 1.77 \times 10^{-8} \text{ ohm-square-meter/meter at } 20°\text{C} \end{aligned}$$

It is this value of ρ which is used in the following two sections where the series resistance is considered as a function of frequency. A discussion of resistance variation with temperature at high frequencies is given in "The Transmission Characteristics of Open-Wire Telephone Lines," by E. I. Green, *B.S.T.J.*, Oct. 1930.

Illustrative Example. Let it be required to determine the resistance per 1000-foot loop of line made up of concentric cylindrical conductors having a No. 14 hard-drawn copper conductor for the inner cylinder and a hard-drawn copper tube of $r_2 = 0.10$ in. and $r_3 = 0.130$ in. for the outer cylinder. Resistance to be determined at an operating temperature of 35°C.

From the wire table the resistance per 1000 feet of No. 14 hard-drawn copper conductor is 2.595 ohms at 20°C.

For the outer conductor use $\rho_2 \dfrac{l}{A_2}$ from equation 1–50.

$$\rho_2 \frac{l}{A_2} = 10.66 \times \frac{1000}{27{,}600} = 0.386 \text{ ohm at } 20°\text{C}$$

where $A_2 = d_3^2 - d_2^2 = 260^2 - 200^2 = 27{,}600$ cir mils

From equation 1–50,

$$R_{\text{loop}} = 2.595 + 0.386 = 2.981 \text{ ohms at } 20°\text{C}$$

To determine the resistance at 35°C use equation 1–51:

$$R_{\text{loop}} = \frac{240 + t_2}{240 + t_1} \times R_1$$

$$= \frac{240 + 35}{240 + 20} \times 2.981 = 3.153 \text{ ohms at } 35°\text{C}$$

8. Change in Resistance and Inductance Because of Skin Effect. When a wire is carrying an alternating current or in general any current whose magnitude is varying, the current has a tendency to crowd toward the surface of the wire. This phenomenon is called *skin effect*. The reason for its existence may be seen from the following elementary analysis. Assume the wire to be divided into a large number of very small circular elements all exactly alike and parallel to the conductor axis. Assume also that the wire is carrying a current which is uniformly distributed throughout the cross section and that this current begins to increase in value. The elements near the axis have more flux encircling them and hence have a higher self-inductance (because of the higher number of flux linkages) than those near the surface. Hence an easier path for current will lie toward the surface of the conductor, and since the material is uniform, the current will seek to flow along the paths of least impedance.

Another concept of skin effect may be explained in terms of RI and XI drops. The drop over any unit length of each of these elements, since they are in parallel, must be the same for all. Therefore, because of the higher inductance and hence higher resultant voltage drop near the center of the wire, there must be higher RI drop near the surface. Since the resistance of all elements is the same, it follows again that the current near the surface must be greater. Hence there must be a redistribution of current throughout the wire when a current of variable magnitude is flowing. An increase in current in the outer portions of the wire will cause a greater heat loss than the reduction due to a decrease of current near the center. This redistribution of current must result in an increase of total conductor resistance since effective resistance is defined in terms of heating effect, which is proportional to current squared.

The correction factor to apply to solid round wires to account for the redistribution of current due to skin effect is shown in Fig. 1–7 where the

ordinate represents the ratio of the effective a-c resistance, R_{ac}, to the zero frequency or d-c resistance, R_{dc}. (The derivation of the results shown in Fig. 1-7 are given in the following sections.)

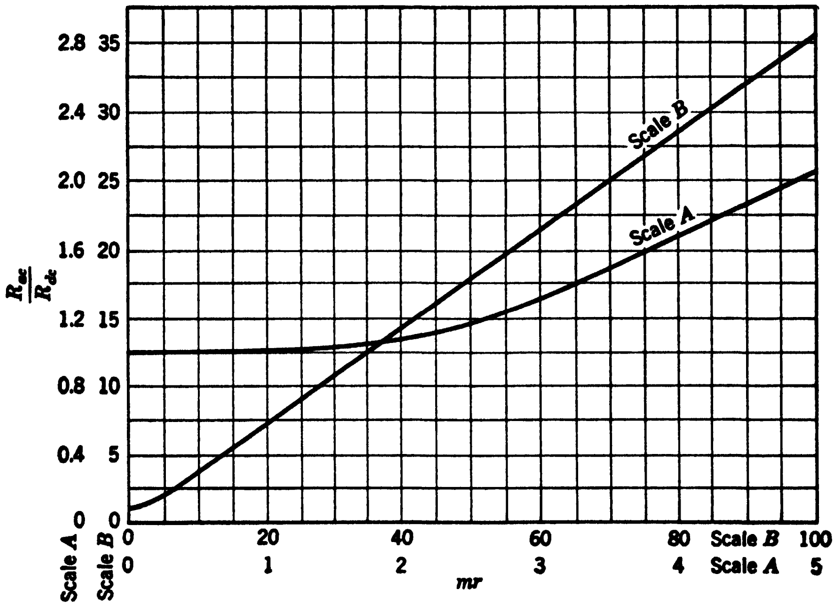


FIG. 1-7. Correction factor for skin effect for solid round wire to be applied to resistance.

The ratio R_{ac}/R_{dc} is a function of the frequency, the conductor material, and the radius of the conductor. These variables are usually grouped together as indicated by mr on the abscissae of Fig. 1-7, where

$$mr = \sqrt{\frac{\omega\mu}{\rho}} \cdot r \quad \text{(in rationalized units)} \qquad [1\text{-}52]$$

and r is the radius of the conductor.

$$m = \sqrt{\frac{\omega\mu}{\rho}}$$

$\omega = 2\pi f$ (radians/sec)

$\mu = \mu_0\mu_r$ (the product of the permeability of free space and the relative permeability of the conductor material)

ρ = resistivity of the conductor material

The product mr may be evaluated in any systematic set of units since it is a dimensionless factor,[7] but, if unrationalized units are employed, it is necessary to use $m = \sqrt{4\pi\omega\mu/\rho}$. The reason for the difference in the forms for m will become evident after the derivation of R_{ac}/R_{dc} is considered. (See Art. 9.)

Illustrative Example. In order to illustrate the use of Fig. 1–7 let it be required to find the resistance of 1000 feet of No. 10 AWG hard-drawn copper conductor at 20°C operating at a frequency of 4 megacycles. (See Appendix VIII for the dimensions and d-c resistance of the conductor.)

In rationalized mks units, the known data are

$$r = \frac{0.1019}{2} \times 0.0254 = 0.001294 \text{ meter}$$

$$\mu = \mu_0 = 4\pi \times 10^{-7} \quad \text{(since } \mu_r \text{ of copper is essentially unity)}$$

$$\rho = 1.77 \times 10^{-8} \text{ ohm-sq m/meter} \quad \text{(See Art. 7.)}$$

$$\omega = 2\pi f = 8\pi \times 10^6 \text{ radians/sec}$$

$$R_{dc} = 1.027 \text{ ohms}$$

hence

$$mr = \left(\sqrt{\frac{8\pi \times 10^6 \times 4\pi \times 10^{-7}}{1.77 \times 10^{-8}}}\right)(1.294 \times 10^{-3}) = 54.7$$

The ratio R_{ac}/R_{dc} for $mr = 54.7$ is found from the B scale of Fig. 1–7 to be 19.6 and $R_{ac} = 19.6 \times 1.027 = 20.1$ ohms.

The same result might be obtained employing any other set of systematic units. In the unrationalized ab-cgs system, for example, where

$$m = \sqrt{\frac{4\pi\omega\mu}{\rho}}$$

$$\mu_0 = 1$$

and

$$\rho = 1.77 \times 10^3 \text{ abohms-sq cm/cm}$$

$$mr = \left(\sqrt{\frac{4\pi \times 8\pi \times 10^6 \times 1}{1.77 \times 10^3}}\right)(0.1294) = 54.7$$

[7] Since μ_0 is dimensionally F^1I^{-2} (where F represents force), μ_r is dimensionless, and ω is dimensionally T^{-1}, it follows that

$$m^1r^1 = \sqrt{\frac{\omega\mu r^2}{\rho}} = \sqrt{\frac{T^{-1}F^1I^{-2}l^2}{R^1l^1}}$$

or, since RI^2 is dimensionally equal to $F^1l^1T^{-1}$,

$$m^1r^1 = \sqrt{\frac{T^{-1}F^1l^1}{R^1I^2}} = \sqrt{\frac{T^{-1}F^1l^1}{F^1l^1T^{-1}}}$$

which is the same as that obtained when rationalized mks units are employed.

The reason for the dual evaluation of mr is to emphasize the fact that $m = \sqrt{\omega\mu/\rho}$ in rationalized units and $m = \sqrt{4\pi\omega\mu/\rho}$ in unrationalized units. Both forms of m appear in the literature.

9. Derivation of R_{ac}/R_{dc}. Consider any cylindrical element of a solid round conductor which has a radius x and an axial length of unity. A cross-sectional view of one such element is shown by the *abdc* rectangle

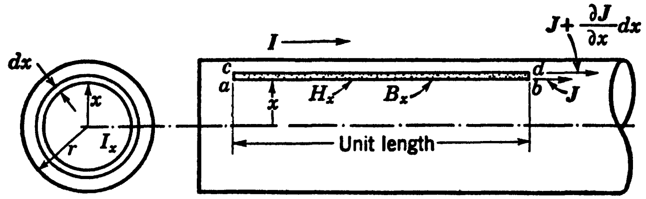


FIG. 1–8. Showing cylindrical element of solid round wire.

in Fig. 1–8. Qualitative reasons for the difference in current density J at radii of x and $(x + dx)$ were given in Art. 8. Mathematically,

J is the current density at a distance x from the center;

$\left(J + \frac{\partial J}{\partial x}\,dx\right)$ is the current density at a distance $(x + dx)$ from the center.

It should be recognized that in general J (or J) is a function of both x and time t.

If the frequency of the alternating current carried by the conductor is fairly high, an appreciable $d\phi/dt$ voltage is generated in the *abdc* loop of Fig. 1–8 which for voltage equilibrium requires unbalanced RI drops along the *ab* and *cd* paths. These RI drops may be written on a unit length basis as

$$R_{ab}I_{ab} = \rho\frac{l}{A}\cdot I_{ab} = \rho J \qquad [1\text{–}53]$$

and

$$R_{cd}I_{cd} = \rho\frac{l}{A}\cdot I_{cd} = \rho\left(J + \frac{\partial J}{\partial x}\,dx\right) \qquad [1\text{–}54]$$

Since

$$\oint \mathcal{E}\cdot dl = -\frac{\partial\phi}{\partial t} \quad \text{(around any closed loop)}$$

we may write

$$R_{ab}I_{ab} - R_{cd}I_{cd} = \rho J - \rho\left(J + \frac{\partial J}{\partial x}\,dx\right) = -\frac{\partial \phi}{\partial t} \qquad [1\text{–}55]$$

or

$$\rho\,\frac{\partial J}{\partial x}\,dx = \frac{\partial B_x}{\partial t}\,dx = \frac{\partial \mu H_x}{\partial t}\,dx$$

from which

$$\rho\,\frac{\partial J}{\partial x} = \mu\,\frac{\partial H_x}{\partial t} \quad \text{(assuming } \mu \text{ constant)} \qquad [1\text{–}56]$$

If rationalized mks units are employed,

$$H_x = \frac{I_x}{2\pi x} \quad \text{ampere-turns/meter} \qquad [1\text{–}57]$$

where I_x is the current enclosed within the πx^2 portion of the conductor area or the current which establishes H_x.

Recognizing that J is a function of x as well as of t,

$$I_x = \int_0^x J2\pi x dx \qquad [1\text{–}58]$$

From equations 1–57 and 1–58,

$$xH_x = \int_0^x Jxdx \qquad [1\text{–}59]$$

If this equation is differentiated with respect to x, there is obtained

$$x\,\frac{\partial H_x}{\partial x} + H_x = Jx$$

or

$$\frac{\partial H_x}{\partial x} + \frac{H_x}{x} = J \qquad [1\text{–}60]$$

which if differentiated with respect to t yields

$$\frac{\partial^2 H_x}{\partial x \partial t} + \frac{1}{x}\cdot\frac{\partial H_x}{\partial t} = \frac{\partial J}{\partial t} \qquad [1\text{–}61]$$

Equation 1–56 rearranged gives

$$\frac{\partial H_x}{\partial t} = \frac{\rho}{\mu}\cdot\frac{\partial J}{\partial x} \qquad [1\text{–}62]$$

and, differentiated with respect to x,

$$\frac{\partial^2 H_x}{\partial t \partial x} = \frac{\rho}{\mu} \cdot \frac{\partial^2 J}{\partial x^2} \quad [1\text{-}63]$$

Substituting the above values for $\partial H_x/\partial t$ and $\partial^2 H_x/\partial x \partial t$ in equation 1–61, an equation in J is obtained

$$\frac{\partial^2 J}{\partial x^2} + \frac{1}{x} \cdot \frac{\partial J}{\partial x} - \frac{\mu}{\rho} \cdot \frac{\partial J}{\partial t} = 0 \quad [1\text{-}64]$$

If the current in the conductor is a sinusoidally time-varying current having an angular frequency of ω, J may for analytical purposes be represented as $J_x \epsilon^{j\omega t}$ where $\epsilon^{j\omega t} = \cos \omega t + j \sin \omega t$ and J_x is a function of x only. Under these conditions equation 1–64 becomes

$$\frac{\partial^2 J_x}{\partial x^2} + \frac{1}{x} \cdot \frac{\partial J_x}{\partial x} - j \frac{\omega \mu}{\rho} J_x = 0 \quad [1\text{-}65]$$

where $j(=\sqrt{-1})$ is used in its customary sense to mean an operator which advances the time phase of a complex quantity through 90°.

Equation 1–65 is a Bessel's equation which defines the current distribution within the conductor, and a solution of this equation will yield the results shown in Fig. 1–7. This equation is customarily written as

$$\frac{\partial^2 J_x}{\partial x^2} + \frac{1}{x} \cdot \frac{\partial J_x}{\partial x} - jm^2 J_x = 0 \quad [1\text{-}66]$$

where $m = \sqrt{\omega \mu / \rho}$, and, since this derivation has been performed in terms of rationalized units, the value of μ to employ is $4\pi \times 10^{-7}$ if the mks system is used.

10. Solution of Equation 1–66. If m in equation 1–66 is assumed to be constant for any specified conductor, a solution of this equation for J_x takes the form:

$$J_x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \cdots \quad [1\text{-}67]$$

$$-jm^2 J_x = -jm^2 a_0 - jm^2 a_1 x - jm^2 a_2 x^2 - jm^2 a_3 x^3 - jm^2 a_4 x^4 - jm^2 a_5 x^5 - jm^2 a_6 x^6 -$$

$$\frac{1}{x} \cdot \frac{\partial J_x}{\partial x} = a_1 \frac{1}{x} + 2a_2 + 3a_3 x + 4a_4 x^2 + 5a_5 x^3 + 6a_6 x^4 + 7a_7 x^5 + 8a_8 x^6 +$$

$$\frac{\partial^2 J_x}{\partial x^2} = + 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + 56a_8 x^6 +$$

If the a's can be evaluated and determined uniquely in terms of the boundary conditions, then the assumed form of J_x will be useful pro-

vided the infinite series converges rapidly. Since a_0 is the value of J_x at $x = 0$, the other a's will be evaluated in terms of a_0. It is to be noted that the above three functions whose sum is always zero (see equation 1–66) have been arranged in columns of $1/x$, constants, x, x^2, etc. With this arrangement a given a can be evaluated in terms of a_0 almost by inspection, as

$$2a_2 + 2a_2 - jm^2a_0 = 0$$

or

$$a_2 = \frac{jm^2a_0}{4}$$

The a's are then seen to be

$$a_1 = 0, \quad a_2 = \frac{jm^2a_0}{4}, \quad a_3 = \frac{jm^2a_1}{9} = 0, \quad a_4 = \frac{jm^2a_2}{16} = -\frac{m^4a_0}{2^4(2!)^2},$$

$$a_5 = \frac{jm^2a_3}{25} = 0, \quad a_6 = \frac{jm^2a_4}{36} = -\frac{jm^6a_0}{2^6(3!)^2}, \quad a_7 = 0,$$

$$a_8 = \frac{jm^2a_6}{64} = \frac{m^8a_0}{2^8(4!)^2}, \qquad a_9 = 0, \quad a_{10} = \frac{jm^2a_8}{100} = \frac{jm^{10}a_0}{2^{10}(5!)^2}, \quad \text{etc.}$$

Hence

$$J_x = a_0\left(1 + j\frac{m^2x^2}{2^2} - \frac{m^4x^4}{2^4(2!)^2} - j\frac{m^6x^6}{2^6(3!)^2} + \frac{m^8x^8}{2^8(4!)^2} + j\frac{m^{10}x^{10}}{2^{10}(5!)^2} - \frac{m^{12}x^{12}}{2^{12}(6!)^2} - j\cdots\right)$$

$$J_x = a_0\underbrace{\left(1 - \frac{m^4x^4}{2^4(2!)^2} + \frac{m^8x^8}{2^8(4!)^2} - \frac{m^{12}x^{12}}{2^{12}(6!)^2} + \cdots\right)}_{(\cdots\cdots\text{ called ber } mx \cdots\cdots)}$$

$$+ ja_0\underbrace{\left(\frac{m^2x^2}{2^2} - \frac{m^6x^6}{2^6(3!)^2} + \frac{m^{10}x^{10}}{2^{10}(5!)^2} - \cdots\right)}_{(\cdots\cdots\text{ called bei } mx \cdots\cdots)} \qquad [1\text{–}68]$$

The series designated by ber mx and that for bei mx are abbreviations for the real and imaginary parts of the Bessel function $J_n(x)$. Note: J is used as a mathematical symbol in the solution of Bessel's equations and here as a current density. The text material should clearly indicate the subject matter under discussion so that no confusion should arise from this dual usage of the symbol J.

11. Evaluation of R_{ac}/R_{dc} and L_{ac}/L_{dc}. The voltage drop along any path like ab or cd in Fig. 1–8 is equal to the voltage drop along the outer

lamination where $x = r$ since the flow of current is directed along the axial length of the conductor. The voltage drop along the outer lamination per unit length of conductor is

$$e_r = \rho J_r = \rho a_0(\text{ber } mr + j \text{ bei } mr) \tag{1-69}$$

This voltage drop divided by the total conductor current I represents the internal impedance of the conductor per unit length. At any instant of time

$$I = \int_0^r J_x 2\pi x dx = 2\pi a_0 \int_0^r x(\text{ber } mx + j \text{ bei } mx)dx \tag{1-70}$$

Employing the series forms shown in equation 1-68 for ber mx and bei mx, it is easily shown by termwise integration that

$$\begin{aligned} I = \frac{2\pi a_0 r}{m}\Bigg[&\left(\frac{mr}{2} - \frac{6m^5r^5}{2^6(3!)^2} + \frac{10m^9r^9}{2^{10}(5!)^2} - \cdots\right) \\ &(\cdots\cdots \text{called bei}' \, mr \cdots\cdots) \\ &+ j\left(\frac{4m^3r^3}{2^4(2!)^2} - \frac{8m^7r^7}{2^8(4!)^2} + \frac{12m^{11}r^{11}}{2^{12}(6!)^2} - \cdots\right)\Bigg] \\ &(\cdots\cdots\cdot \text{called } -\text{ber}' \, mr \cdots\cdots\cdot) \end{aligned} \tag{1-71}$$

where bei′ mr is the first derivative of bei mr, and ber′ mr is the first derivative of ber mr, as may be readily checked by differentiating bei mr and ber mr with respect to mr.

The internal impedance per unit length of conductor is

$$Z_i = \frac{e_r}{I} = \frac{\rho m}{2\pi r} \cdot \left(\frac{\text{ber } mr + j \text{ bei } mr}{\text{bei}' \, mr - j \text{ ber}' \, mr}\right) \tag{1-72}$$

Rationalizing gives

$$Z_i = \frac{\rho m}{2\pi r} \times \left[\frac{(\text{ber } mr \text{ bei}' \, mr - \text{bei } mr \text{ ber}' \, mr) + j(\text{bei } mr \text{ bei}' \, mr + \text{ber } mr \text{ ber}' \, mr)}{\text{bei}'^2 \, mr + \text{ber}'^2 \, mr}\right] \tag{1-73}$$

The real part of equation 1-73 is interpreted to be the effective series resistance of the conductor and the j part as the internal self-inductance of the conductor per unit length. Thus

$$Z_i = R_{ac} + j\omega L_{ac} \quad \text{(per unit length)} \tag{1-74}$$

Since

$$R_{dc} = \frac{\rho}{\pi r^2} \quad \text{on a per unit length basis,}$$

$$\frac{R_{ac}}{R_{dc}} = \frac{mr}{2}\left(\frac{\text{ber}\, mr\, \text{bei}'\, mr - \text{bei}\, mr\, \text{ber}'\, mr}{\text{bei}'^2\, mr + \text{ber}'^2\, mr}\right) \qquad [1\text{–}75]$$

Since the d-c internal self-inductance is $\mu/8\pi$ units of inductance per unit length of conductor (see equation 1–2) and since the internal inductive reactance is

$$\omega L_{ac} = \frac{\rho m}{2\pi r}\left(\frac{\text{bei}\, mr\, \text{bei}'\, mr + \text{ber}\, mr\, \text{ber}'\, mr}{\text{bei}'^2\, mr + \text{ber}'^2\, mr}\right) \qquad [1\text{–}76]$$

then

$$\frac{L_{ac}}{L_{dc}} = \frac{4}{mr}\left(\frac{\text{bei}\, mr\, \text{bei}'\, mr + \text{ber}\, mr\, \text{ber}'\, mr}{\text{bei}'^2\, mr + \text{ber}'^2\, mr}\right) \qquad [1\text{–}77]$$

By means of equation 1–75, the ratio R_{ac}/R_{dc} may be evaluated for various values of mr and plotted as shown in Fig. 1–7.

By means of equation 1–77, the ratio L_{ac}/L_{dc} may be evaluated for various values of mr and plotted as shown in Fig. 1–9.

Illustrative Example. In order to illustrate the use of equations 1–75 and 1–77 let it be required to determine these ratios for an mr value of 3. Ber mr and bei mr are evaluated from the respective series given in equation 1–68 and ber′ mr and bei′ mr from the series given in equation 1–71.

$$\text{ber}\, mr = \text{ber}\, 3 = -0.2214$$

$$\text{bei}\, mr = \text{bei}\, 3 = 1.9376$$

$$\text{ber}'\, mr = \text{ber}'\, 3 = -1.5699$$

$$\text{bei}'\, mr = \text{bei}'\, 3 = 0.8805$$

From equation 1–75,

$$\frac{R_{ac}}{R_{dc}} = 1.5\left[\frac{(-0.2214)(0.8805) - (1.9376)(-1.5699)}{0.8805^2 + 1.5699^2}\right] = 1.318$$

It will be observed that this ratio agrees with that shown in Fig. 1–7 and indicates that the a-c resistance is 1.318 times the d-c resistance. From equation 1–77,

$$\frac{L_{ac}}{L_{dc}} = 1.33\left[\frac{(1.9376)(0.8805) + (-0.2214)(-1.5699)}{0.8805^2 + 1.5699^2}\right] = 0.845$$

This result agrees with that shown in Fig. 1–9 and indicates that the internal self-inductance of a conductor (operating under conditions such that $mr = 3$) is 0.845 times the low-frequency or d-c internal inductance.

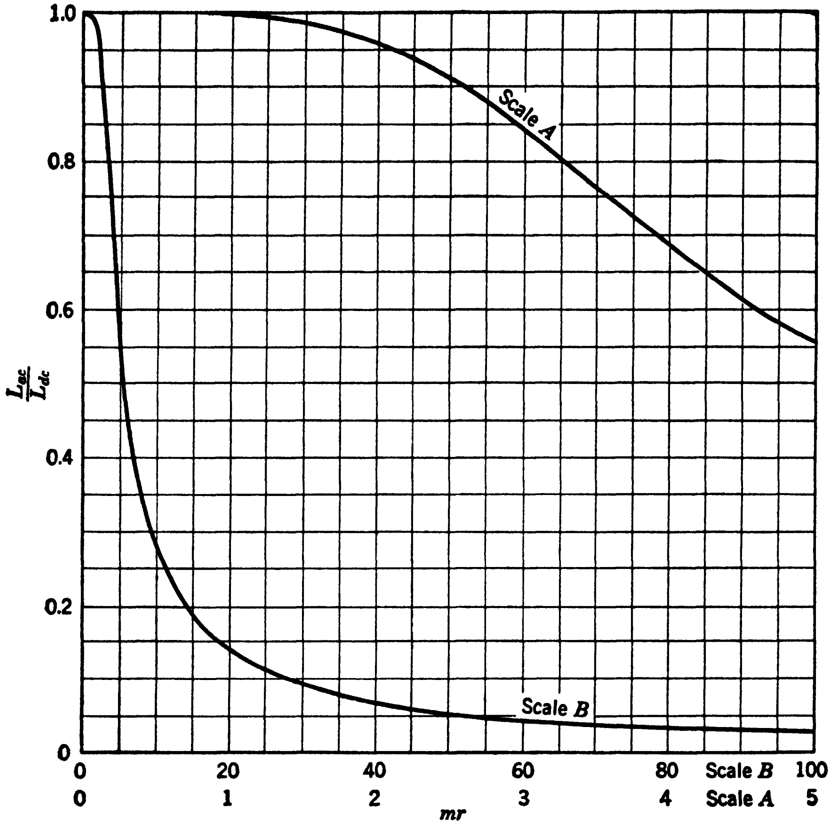


FIG. 1-9. Correction factor for skin effect for solid round wire to be applied to inductance. Multiply the term due to internal flux linkage by the ratio L_{ac}/L_{dc}.

The value of $mr = 3$ may of course be obtained in a variety of ways. The value of m for a copper wire carrying a current of 10,000 cycles per second is

$$m = \sqrt{\frac{\omega\mu}{\rho}} = \sqrt{\frac{2\pi \times 10^4 \times 4\pi \times 10^{-7}}{1.77 \times 10^{-8}}} = 2115$$

For $mr = 3$,

$$r = \frac{3}{2115} = 0.00142 \text{ meter} \quad \text{or} \quad 0.142 \text{ cm}$$

The same value of mr would apply to a copper conductor which is half the diameter and carries a current of 40,000 cycles per second.

12. Approximate Expressions for R_{ac} at Very High Frequencies. At very high frequencies, the a-c resistance of a conductor of ordinary size

is approximately the same as the d-c resistance of a hollow conductor having the same outer dimension[8] and a thickness of

$$\delta = 503\sqrt{\frac{\rho}{f\mu_r}} \quad \text{meter} \qquad [1\text{–}78]$$

where ρ is the resistivity in ohm-square-meters per meter,
f is the frequency in cycles per second,
μ_r is the relative permeability of the conductor material.

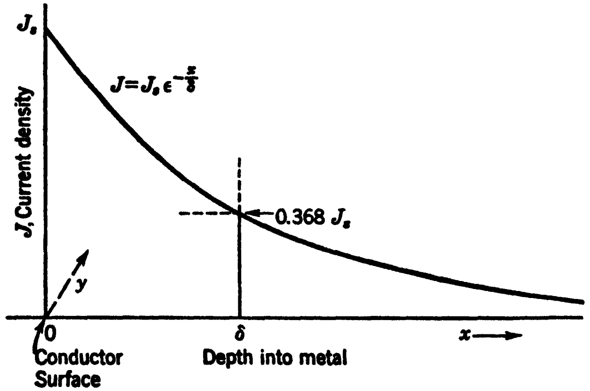


FIG. 1–10. Exponential decay of current from surface of conductor.

Equation 1–78 is called the depth of penetration formula, and it may be used to derive approximate expressions for the a-c resistance of either parallel-wire lines or coaxial cables. As applied to a solid round conductor, it is evident that the effective area of the equivalent hollow conductor is

$$A_\delta = \pi r^2 - \pi(r - \delta)^2 = 2\pi r\delta - \pi\delta^2 \qquad [1\text{–}79]$$

[8] After Maxwell's equations have been considered, it can be shown that, as a good approximation, the current density decays exponentially in magnitude from the surface of the conductor, as indicated in Fig. 1–10. (See Appendix VI.) The total current in a conductor of any fixed width y is

$$I = \int_0^\infty J(y\,dx) = y\int_0^\infty J_s\epsilon^{-x/\delta}dx = yJ_s[-\delta\epsilon^{-x/\delta}]_0^\infty = y\delta J_s$$

where J_s is the current density at the surface.

Thus the total current I may be taken into account by the simple expedient of assuming a uniform current density of J_s over the entire depth of penetration δ. By means of this assumption the high-frequency resistance is approximated as

$$R_{ac} = R_{dc}$$

provided R_{dc} is evaluated as $\rho l/y\delta$, where y is the periphery of the conductor.

If δ is very small compared to r, the a-c resistance of the conductor may be written as

$$R_{ac} = \frac{\rho}{A_\delta} \doteq \frac{\rho}{2\pi r\delta} = \frac{1}{2\pi \times 503} \cdot \frac{\sqrt{f\mu_r\rho}}{r_m} \text{ ohms/meter of single conductor} \qquad [1\text{–}80]$$

where r_m indicates that the radius of the conductor is to be expressed in meters.

Convenient working expressions for R_{ac} may be obtained directly from equation 1–80. In transmission-line work it is often desirable to have R_{ac} of a copper line expressed in ohms per loop meter and in ohms per loop mile. In this case $\rho = 1.77 \times 10^{-8}$ ohm-sq-m per meter and $\mu_r = 1$. Inserting these values into equation 1–80 and expressing the radius in centimeters rather than in meters, there is obtained for parallel hard-drawn copper conductors,

$$R_{ac} = 84.2 \frac{\sqrt{f}}{r_{cm}} \times 10^{-7} \text{ ohm/loop meter} \qquad [1\text{–}81]$$

or

$$R_{ac} = 135 \frac{\sqrt{f}}{r_{cm}} \times 10^{-4} \text{ ohm/mile of line} \qquad [1\text{–}82]$$

where r_{cm} is the radius of the conductor in centimeters and f is the transmitted frequency in cycles per second.

For a coaxial line composed of hard-drawn copper conductors,

$$R_{ac} = 42.1\sqrt{f}\left(\frac{1}{a} + \frac{1}{b}\right) \times 10^{-7} \text{ ohm/meter} \qquad [1\text{–}83]$$

or

$$R_{ac} = 67.5\sqrt{f}\left(\frac{1}{a} + \frac{1}{b}\right) \times 10^{-4} \text{ ohm/mile} \qquad [1\text{–}84]$$

where a is the radius of the inner conductor, b is the inner radius of the outer conductor, each in centimeters, and f is the transmitted frequency in cycles per second.

In using equations 1–80 through 1–84 at intermediate frequencies (those in the order of 100,000 cycles per second) one may estimate the degree of approximation involved by noting the magnitude of δ^2 relative to $2r\delta$ where δ is evaluated by means of equation 1–78. It will be remembered that $\pi\delta^2$ of equation 1–79 was considered to be negligible relative to $2\pi r\delta$ in arriving at equation 1–80.

Illustrative Example. In order to emphasize the limitations of equations 1–81 through 1–84, let it be required to find the ratio R_{ac}/R_{dc} of a copper

conductor 0.142 centimeter in radius operating at 10,000, 1,000,000, and 10,000,000 cycles per second by means of equation 1–81 and compare the results with those given in Fig. 1–7. The latter values are the more precise since they are based on equation 1–75.

$$R_{dc} = \rho \frac{l}{A} = 1.77 \times 10^{-8} \times \frac{2}{\pi(0.00142)^2}$$

$$= 0.00559 \text{ ohm/2-meter length of wire (loop meter of line)}$$

Employing equation 1–81, at 10,000 cycles per second, gives

$$R_{ac} = \frac{84.2 \times \sqrt{10{,}000} \times 10^{-7}}{0.142} = 0.00593 \text{ ohm/loop meter}$$

$$\frac{R_{ac}}{R_{dc}} = \frac{0.00593}{0.00559} = 1.06$$

The value of mr of this conductor operating at 10,000 cycles per second is

$$mr = \sqrt{\frac{\omega\mu}{\rho}} \cdot r = \sqrt{\frac{2\pi \times 10^4 \times 4\pi \times 10^{-7}}{1.77 \times 10^{-8}}} \times (0.00142) = 3$$

From Fig. 1–7,

$$\frac{R_{ac}}{R_{dc}} = 1.32 \quad \text{at } mr = 3$$

For this case, from equation 1–78,

$$\delta = 503 \sqrt{\frac{1.77 \times 10^{-8}}{10^4 \times 1}} = 0.000669 \text{ meter} \quad \text{or} \quad 0.0669 \text{ cm}$$

and δ_{cm}^2 (= 0.00447) is not negligible relative to $2r\delta_{cm}$ (= 0.019).

The results of the calculations for the other frequencies are made in a similar manner, and all are given in the following table for comparison.

f cycles/sec	R_{ac}/R_{dc} from eq. 1–81	mr	R_{ac}/R_{dc} from Fig. 1–7	δ_{cm}^2	$2r\delta_{cm}$	$\frac{\delta_{cm}^2}{2r\delta_{cm}}$
10^4	1.06	3	1.32	0.00447	0.019	0.235
10^6	10 6	30	10.9	0.0000447	0.0019	0.0235
10^7	33.5	94.8	33.8	0 00000447	0 0006	0.0075

As shown in the table, the correspondence becomes better as the value of δ^2 becomes smaller relative to $2r\delta$.

13. Shunt Conductance. When two parallel conductors have a potential difference between them, a certain amount of current will flow because of the finite resistance of the insulation. For a line mounted on glass insulators this leakage may be very small in dry weather. For a cable where the wires are close together and the insulation is not too good, or perhaps damp, the leakage may be rather large. Since we

customarily speak in terms of units of 1 mile of line, it is usual to state conductance as a certain number of mhos (or micromhos) per mile. The phenomenon of leakage on a line is analogous to having a large number of very high-resistance wires, uniformly spaced along the line, connected between the two wires. Conductance is usually treated as all lumped at one place in each section of a line or uniformly distributed along the line. Actually it may be neither, but any error due to this discrepancy is entirely negligible. Leakage conductance will be denoted by the letter G, and it is in general specified on a per-unit-length-of-line basis if it has a significant effect on the operation of the line; otherwise it is considered to be zero.

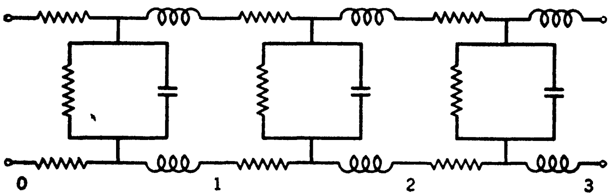


FIG. 1–11. Representation of a transmission line.

14. The Short Line with Lumped Constants. We have seen that a line has distributed along its entire length a great number of infinitesimal series resistors, coils, capacitors, and shunt resistors. The consideration of these units to the present time is sufficient to tell us how these are arranged. With every element of length along the line, no matter how small, there is associated a small series resistance R, an inductance L, a shunt capacitance C, and a shunting resistor whose value of conductance we have designated as G $(= 1/R_{sh})$. Thus it is clear that, if accuracy is desired in the schematic representation of a line, the line should be drawn somewhat as shown in Fig. 1–11, where the length of each section, as 0 to 1, 1 to 2, etc., is infinitesimal. It thus appears that a diagram of the communication line cannot be drawn exactly correctly. Furthermore, for the purposes of calculation according to elementary circuit methods, such a circuit does not lead to exceptional simplicity.

The next best compromise, and one which immediately suggests itself, is to employ, not an infinitesimal section, but a section of finite length. It is clear that, if one uses as a section the entire line of perhaps many miles and places the shunt capacitance and conductance across the center, the results would be considerably in error. Actually current is leaking across the line all the way from the generator, and

thus, as we proceed along the line to the center, the current in the wires is progressively different, and the elementary inductance and resistance near the center are not carrying the same current which started from the generator terminals. However, in placing the capacitance and conductance across the center of the line, the assumption would be made that no change in current occurs until the current arrives at the center. Thus a compromise must be made between a long section which is too inaccurate and a very short section. Since all sections can be considered alike, the characteristics of one short section can be calculated and the results extended to include the required number of sections necessary to make up the entire line.

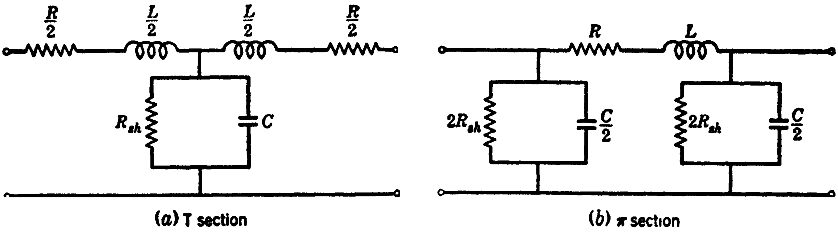


FIG. 1-12. Representation of a transmission line.

Thus it is possible to settle on a relatively short section as representing the line provided that the total capacitance across the line, or any other parameter, is made equal to the measured or calculated amount for that length. This statement does not imply that the capacitance and leakage conductance must be across the center of the section. Indeed they may be divided with a part placed at one end and the remainder at the other end. In other words, both the sections shown in Fig. 1-12 are permissible since these two sections have the same total L, C, R, and G. However, if calculations are made using these two sections while employing a given generator voltage at one end and load resistor across the other, the same results will not be obtained, nor will either answer be correct. Frequently the results obtained using either type of section are sufficiently good for practical purposes. Work on transmission lines thus usually begins with a consideration of these relatively simple components — the elementary sections.

PROBLEMS

(All copper conductors specified in these problems are assumed to be hard drawn and operating at 20°C, unless otherwise noted.)

1-1. Given a line constructed of two parallel No. 000 conductors spaced 15 inches center to center, find the low-frequency self-inductance in henrys of 5 miles of line.

1-2. Find the self-inductance per loop meter in microhenrys of a parallel-wire line operating at a very high frequency (where the internal inductance of the conductor is negligibly small) if the line is made of copper tubing having an outside diameter of $\frac{1}{4}$ inch separated center to center by a distance of 6 inches.

1-3. Find the low-frequency inductance per meter length of an unorthodox parallel-wire transmission line which is composed of one No. 10 iron wire and one No. 10 copper wire. The center-to-center separation is 12 inches, and the average relative permeability of the iron wire is 100.

1-4. Calculate the self-inductance of the inner conductor of a coaxial arrangement when carrying low-frequency currents if $r_1 = \frac{1}{8}$, $r_2 = 1\frac{1}{2}$, and $r_3 = 1\frac{5}{8}$, all in inches. Note that the units of r_1, r_2, and r_3 are immaterial, provided all units employed are identical. Nonferromagnetic materials are employed.

Calculate the self-inductance of the outer conductor of this coaxial arrangement.

Express the total self-inductance in henrys per meter and in millihenrys per mile.

Calculate the self-inductance of this coaxial arrangement when very high-frequency currents are carried. Compare with the total self-inductance obtained when low-frequency currents are carried.

1-5. Repeat Prob. 1-4 for a coaxial arrangement wherein $r_1 = \frac{1}{32}$, $r_2 = \frac{1}{8}$, and $r_3 = \frac{5}{32}$, all in inches.

1-6. The radius of the inner conductor of a coaxial cable is 0.1 centimeter, and the inner radius of the outer conductor is 0.4 centimeter. Find the low-frequency inductance in millihenrys per mile of cable neglecting the flux linkages in the outer conductor.

1-7. Assume that a metal tape having ferromagnetic properties and a thickness of 0.02 centimeter is spiraled around the inner conductor of the coaxial cable of Prob. 1-6 to form a single layer. (The tape is insulated from the inner conductor.) The relative permeability of the tape is assumed to be constant at a value of 50. Find the self-inductance in millihenrys per mile of cable.

1-8. A line is constructed of two parallel No. 000 copper conductors spaced 15 inches center to center. Find the capacitance of the line in microfarads per mile.

1-9. Find the capacitance in micromicrofarads per loop meter of the parallel-wire line described in Prob. 1-2. $\epsilon_r = 1$.

1-10. Find the capacitance in microfarads per loop mile of the coaxial cable described in Prob. 1-6 if a dielectric material having a relative permittivity of 5 is employed between the inner and outer conductors.

1-11. Find the capacitance in micromicrofarads per loop meter of a parallel-wire line consisting of two No. 0000 AWG copper wires spaced 1.38 inches center to center. $\epsilon_r = 1$.

1-12. The conductors of Fig. 1-6 have a center-to-center separation of 4 centimeters and a radius of 1 centimeter. Calculate the separation of the two equivalent line charges which may be used to replace the distributed charges on the two conductors with due regard for the proximity effect.

1-13. What is the a-c resistance of a conductor operating under conditions such that $mr = 20$ if the d-c resistance is known to be 0.6 ohm?

1-14. What is the ratio R_{ac}/R_{dc} of an aluminum conductor having a resistivity 1.6 times that of hard-drawn copper if the operating frequency is 500 kilocycles per second? The radius of the conductor is 0.3 centimeter.

1–15. Evaluate mr for a hard-drawn copper wire 0.482 inch in diameter at 60 cycles per second in rationalized mks units.

1–16. If the current density at the center of the conductor given in Prob. 1–15 is $1000\underline{/0°}$ amperes per square inch, find the current density at the surface of the conductor in complex polar form.

1–17. (*a*) Evaluate ber mr, bei mr, ber$'$ mr, and bei$'$ mr for $mr = 2$, employing only the first two terms of the respective series forms of these functions.

(*b*) Compare the values of ber 2 and bei 2 found above with the following values which are correct to four significant figures.

$$\text{ber } 2 = 0.7517 \qquad \text{bei } 2 = 0.9723$$

1–18. Evaluate the ratio R_{ac}/R_{dc} when $mr = 2$, using the first two terms of the respective series for the ber, bei, ber$'$, and bei$'$ functions.

1–19. Determine the ratio of the internal self-inductance to the internal d-c self-inductance of a conductor operating under conditions such that $mr = 2$, employing ber, bei, ber$'$, and bei$'$ functions.

1–20. A parallel-wire line is composed of two No. 10 AWG hard-drawn copper conductors separated center to center by a distance of 5 inches. The length of the line is 100 meters.

(*a*) Determine the low-frequency inductance of the line.

(*b*) Determine the 1000-kilocycle inductance of the line.

1–21. What is the ratio of $\omega L/R$ at a frequency of 1000 kilocycles per second of the line described in Prob. 1–20?

1–22. Two parallel copper conductors are 0.5 inch in diameter and are spaced 2.50 inches center to center. The frequency to be transmitted is 580,000 cycles per second. Find the resistance of a mile of the line at this frequency.

1–23. What is the resistance per meter length of a hard-drawn copper conductor which is elliptical in cross section if the outer periphery is 0.5 inch and the operating frequency is 100 megacycles per second? The operating temperature is 60°C.

1–24. Find the resistance per meter length of a copper conductor of circular cross section which has a periphery of 0.5 inch and which is operating at a frequency of 100 megacycles per second. The operating temperature is 60°C.

1–25. Given a copper coaxial line having the following dimensions:

Diameter of the inner conductor	= 0.2 cm
Inner diameter of the outer conductor	= 0.8 cm
Outer diameter of the outer conductor	= 1.0 cm

Find the resistance per mile of cable if the operating frequency is 36 megacycles per second.

CHAPTER II

NETWORKS, T AND π SECTIONS

In the previous chapter it was made reasonably clear that a part of a transmission line may be represented by a suitable T or π section. In general, it is advisable to be able to reduce any network to some standard form such as a T or π section and to use on it certain short-cut methods which often greatly simplify the work involved. This chapter, which will treat certain transformations of the elementary T and π sections, will begin with a rather general discussion.

15. Importance of Network Theorems. In the ordinary simple circuit problem, such as that involving a resistance connected across a cell which has a certain emf and internal resistance, the equation representing Ohm's Law can be easily used. As circuits become more and more complex, one begins to wonder if there should not exist relationships or equations which could handle whole groups of minor ideas in somewhat the same manner that Ohm's Law handles the basic ideas of resistance, emf, etc., or Kirchhoff's Laws handle the elements of more general circuits. The analogy is not exact by any means; nevertheless it is useful to keep in mind.

Network theorems provide certain short cuts in the calculation of circuits. It is often possible by means of them to reduce a relatively difficult problem to a simple one.

16. 3-Terminal Networks, T and π Sections. The letter Z with suitable subscript will be assigned to represent the impedance of an elementary R–L–C series circuit where any of the three parameters may or may not be missing. Thus in general

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Any circuit or network, then, will be considered as made up of a number of Z's connected in a suitable manner. Such a circuit with three terminals may be represented as in Fig. 2–1, and elementary T and π sections can always be found in it. It is possible to reduce each ele-

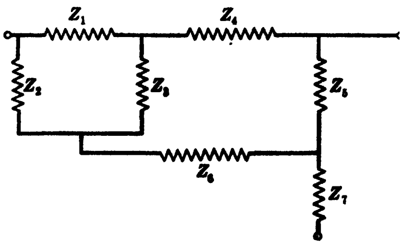
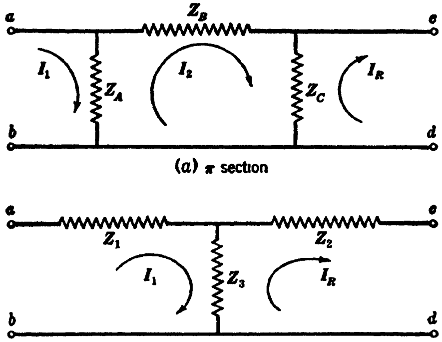


FIG. 2–1. Network.

mentary π section to a T section and then recombine all series impedances, obtaining a new mesh. It is possible again to reduce the resulting π sections to T sections, and such a procedure would eventually reduce the entire circuit of Fig. 2-1 to one T section.

17. Transformation From π to T Section and Vice Versa. Let it be required to find the T section which is equivalent to a given π section (at one frequency). The given π section is as shown in Fig. 2-2*a*. The



(*a*) π section

(*b*) T section

FIG. 2-2. Transformation from π to T section.

circuit required is that shown in Fig. 2-2*b*. If these circuits are to be equivalent, measurements made between the terminals *a*, *b*, *c*, and *d* must be the same for both circuits. Since there are three impedances, only three measurements are needed. Let these be the impedance measurements Z_{ab}, Z_{ac}, and Z_{cd}, and for each measurement let all other terminals be disconnected. Then for both circuits

$$Z_{ab} = \frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C} = Z_1 + Z_3 \qquad [2\text{-}1]$$

$$Z_{ac} = \frac{Z_B(Z_A + Z_C)}{Z_A + Z_B + Z_C} = Z_1 + Z_2 \qquad [2\text{-}2]$$

$$Z_{cd} = \frac{Z_C(Z_A + Z_B)}{Z_A + Z_B + Z_C} = Z_2 + Z_3 \qquad [2\text{-}3]$$

These equations can be solved for Z_1, Z_2, and Z_3 in terms of Z_A, Z_B, and Z_C, or vice versa, provided $Z_A + Z_B + Z_C \neq 0$. First solve for Z_1, Z_2, and Z_3. Add equations 2-1 and 2-2 and subtract equation 2-3:

$$2Z_1 + Z_2 + Z_3 - Z_2 - Z_3 = \frac{Z_AZ_B + Z_AZ_C + Z_AZ_B + Z_BZ_C - Z_AZ_C - Z_BZ_C}{Z_A + Z_B + Z_C}$$

or

$$2Z_1 = \frac{2Z_AZ_B}{Z_A + Z_B + Z_C}$$

$$Z_1 = \frac{Z_AZ_B}{Z_A + Z_B + Z_C} \qquad [2\text{–}4]$$

Going through the same procedure for the other combinations of equations 2–1, 2–2, and 2–3 yields

$$Z_2 = \frac{Z_BZ_C}{Z_A + Z_B + Z_C} \qquad [2\text{–}5]$$

and

$$Z_3 = \frac{Z_CZ_A}{Z_A + Z_B + Z_C} \qquad [2\text{–}6]$$

Thus equations 2–4, 2–5, and 2–6 can be used for reducing any π section to an equivalent T section.

In order to obtain the equations for the inverse transformation the following procedure is suggested. From equations 2–4, 2–5, and 2–6 is obtained

$$Z_1Z_2 + Z_2Z_3 + Z_3Z_1 = \frac{Z_AZ_B^2Z_C + Z_BZ_C^2Z_A + Z_CZ_A^2Z_B}{(Z_A + Z_B + Z_C)^2}$$

$$= \frac{Z_AZ_BZ_C}{Z_A + Z_B + Z_C} \qquad [2\text{–}7]$$

Dividing equation 2–7 by equation 2–4, and then by equations 2–5 and 2–6 in succession gives

$$Z_C = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1} \qquad [2\text{–}8]$$

$$Z_A = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2} \qquad [2\text{–}9]$$

$$Z_B = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3} \qquad [2\text{–}10]$$

Thus transformation in either direction can be made, provided, as stated above, $Z_A + Z_B + Z_C \neq 0$. It is necessary now only to show that the two circuits are equivalent. Connect a generator across terminals *a–b* in Figs. 2–2*a* and 2–2*b* and attach an impedance Z_R across terminals *c–d*. In solving these two circuits Kirchhoff's Laws and

determinants will be used as a review of the methods. In the π section of Fig. 2–2a:

$$E = Z_A(I_1 - I_2)$$
$$0 = Z_A(I_2 - I_1) + Z_B I_2 + Z_C(I_2 - I_R)$$
$$0 = Z_C(I_R - I_2) + Z_R I_R$$

where E is the generator voltage,
I_1 is the generator current,
I_2 is the current through the impedance Z_B,
I_R is the current through Z_R.

$$\begin{aligned} E &= I_1 Z_A \quad - I_2 Z_A \quad + 0 \\ 0 &= -I_1 Z_A \quad + I_2(Z_A + Z_B + Z_C) - I_R Z_C \\ 0 &= 0 \quad - I_2 Z_C \quad + I_R(Z_R + Z_C) \end{aligned}$$

Thus

$$\Delta = \begin{vmatrix} Z_A & -Z_A & 0 \\ -Z_A & (Z_A + Z_B + Z_C) & -Z_C \\ 0 & -Z_C & Z_R + Z_C \end{vmatrix}$$

$$= Z_A(Z_B Z_R + Z_B Z_C + Z_C Z_R)$$

and

$$I_R = \frac{\begin{vmatrix} Z_A & -Z_A & E \\ -Z_A & (Z_A + Z_B + Z_C) & 0 \\ 0 & -Z_C & 0 \end{vmatrix}}{\Delta}$$

$$= \frac{E(Z_A Z_C)}{\Delta} = \frac{E Z_A Z_C}{(Z_B Z_R + Z_B Z_C + Z_C Z_R) Z_A}$$

$$= \frac{E Z_C}{Z_B Z_R + Z_B Z_C + Z_C Z_R} \qquad [2\text{–}11]$$

From Fig. 2–2b

$$E = I_1 Z_1 + Z_3(I_1 - I_R)$$
$$0 = Z_3(I_R - I_1) + (Z_2 + Z_R) I_R$$

or

$$E = I_1(Z_1 + Z_3) - I_R Z_3$$
$$0 = -I_1 Z_3 + I_R(Z_3 + Z_2 + Z_R)$$

$$\Delta = \begin{vmatrix} Z_1 + Z_3 & -Z_3 \\ -Z_3 & Z_3 + Z_2 + Z_R \end{vmatrix}$$

$$= Z_1 Z_3 + Z_2 Z_1 + Z_1 Z_R + Z_3 Z_2 + Z_3 Z_R$$

and

$$I_R = \frac{\begin{vmatrix} Z_1 + Z_3 & E \\ -Z_3 & 0 \end{vmatrix}}{\Delta}$$

$$= \frac{EZ_3}{Z_1Z_3 + Z_2Z_1 + Z_1Z_R + Z_2Z_3 + Z_3Z_R} \qquad [2\text{–}12]$$

Divide numerator and denominator of equation 2–11 by Z_C. Then

$$I_R = \frac{E}{\dfrac{Z_RZ_B}{Z_C} + Z_B + Z_R} \qquad [2\text{–}11a]$$

Substitute equations 2–8 and 2–10 into equation 2–11*a*,

$$I_R = \frac{E}{\dfrac{Z_RZ_1}{Z_3} + \dfrac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3} + Z_R}$$

$$= \frac{EZ_3}{Z_RZ_1 + Z_1Z_2 + Z_2Z_3 + Z_3Z_1 + Z_3Z_R}$$

which is identical with equation 2–12. It can also be shown that the input currents are the same for the two circuits.

The actual procedure in making transformations is often somewhat involved. In order to clarify the problem, one of the possible procedures for changing the circuit of Fig. 2–3*a* into its equivalent T section will be given.

(*a*) Change the π section Z_a, Z_b, Z_c into a T section using equations 2–4, 2–5, and 2–6.

(*b*) The network becomes that shown in Fig. 2–3*b*, where Z_1, Z_2, Z_3 are the elements of the new T section.

(*c*) Change the π section Z_e, Z_f, Z_g into a T section as shown in Fig. 2–3*c* and combine Z_2 and Z_d into a single impedance Z_{2d}.

(*d*) Change the π section Z_h, Z_i, Z_j into a T section as shown in Fig. 2–3*d* and combine Z_3 and Z_4 into a single impedance Z_{34}.

(*e*) Combine Z_5 and Z_7 into one impedance Z_{57} and change the π section Z_6, Z_{57}, Z_9 into a T section as shown in Fig. 2–3*e*.

(*f*) Combine impedances Z_{34}, Z_{10} into one impedance Z_{3410}. Combine impedances Z_8, Z_{11} into one impedance Z_{811}, and change the π section Z_{3410}, Z_{2d}, Z_{811} into a T section as shown in Fig. 2–3*f*.

(*g*) By adding Z_1 and Z_{13}, Z_{12} and Z_{15}, a T section is immediately obtained which will be the final result of the series of transformations.

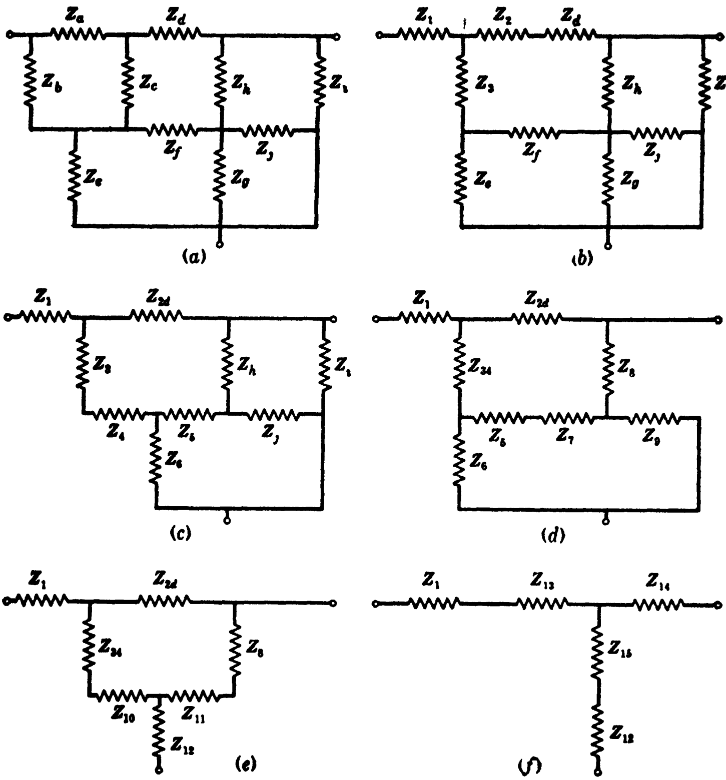


FIG. 2-3. Network reduction.

18. T and π Sections Determined from Impedance Measurements. In the preceding articles it has been shown that T or π sections are fundamental to any discussion of electrical communication networks. Let the problem be proposed to find the equivalent T section if there is given a box with four terminals, inside of which is a circuit connected in any manner whatever to these terminals. The terminals are marked in pairs, " input " and " output."

The easily measured quantities are the open- and short-circuit impedances at each pair of terminals. If it is possible to obtain Z_1, Z_2, and Z_3 in terms of these impedances, then equivalent T (and π) sections may be easily set up. Let the two equivalent circuits be as shown in Fig. 2-4.

Input impedance with output open, $Z_{abo} = Z_1 + Z_3$ [2-13]

Input impedance with output shorted, $Z_{abs} = Z_1 + \dfrac{Z_2 Z_3}{Z_2 + Z_3}$ [2-14]

Output impedance with input open, $Z_{cdo} = Z_2 + Z_3$ [2–15]

Output impedance with input shorted, $Z_{cds} = Z_2 + \dfrac{Z_1 Z_3}{Z_1 + Z_3}$ [2–16]

Subtracting equation 2–13 from equation 2–14

$$Z_{abs} - Z_{abo} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} - Z_1 - Z_3 = \frac{-Z_3^2}{Z_2 + Z_3}$$

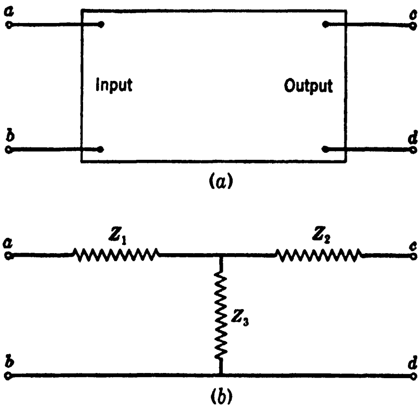


FIG. 2–4. Four-terminal network represented as a T section.

Multiplying by equation 2–15

$$Z_{cdo}(Z_{abs} - Z_{abo}) = -Z_3^2$$

or

$$Z_3 = \pm\sqrt{Z_{cdo}(Z_{abo} - Z_{abs})} \qquad [2\text{–}17]$$

Then from equation 2–13

$$\begin{aligned} Z_1 &= Z_{abo} - Z_3 \\ &= Z_{abo} \mp \sqrt{Z_{cdo}(Z_{abo} - Z_{abs})} \end{aligned} \qquad [2\text{–}18]$$

and from equation 2–15

$$\begin{aligned} Z_2 &= Z_{cdo} - Z_3 \\ &= Z_{cdo} \mp \sqrt{Z_{cdo}(Z_{abo} - Z_{abs})} \end{aligned} \qquad [2\text{–}19]$$

Having found the equivalent T section, the equivalent π section can be found from equations 2–8, 2–9, and 2–10; or a set of equations

giving Z_A, Z_B, and Z_C in terms of the open- and short-circuit impedances can be derived.

There are four equations (2–13, 2–14, 2–15, and 2–16), one of which was not used. It is of interest to express this one in terms of the others. Substituting equations 2–17, 2–18, and 2–19 into equation 2–16

$$Z_{cds} = Z_{cdo} \mp \sqrt{Z_{cdo}(Z_{abo} - Z_{abs})} + \frac{[Z_{abo} \mp \sqrt{Z_{cdo}(Z_{abo} - Z_{abs})}]\,[\pm \sqrt{Z_{cdo}(Z_{abo} - Z_{abs})}]}{Z_{abo}}$$

$$= \frac{Z_{cdo}Z_{abs}}{Z_{abo}} \qquad [2\text{–}20]$$

which can be written

$$\frac{Z_{cds}}{Z_{cdo}} = \frac{Z_{abs}}{Z_{abo}} \qquad [2\text{–}21]$$

19. Simple 4-Terminal Networks. In Art. 17 it was shown that a 3-terminal network may be reduced, at a single frequency, to an equivalent T or π section. Can the same be said of a 4-terminal network?

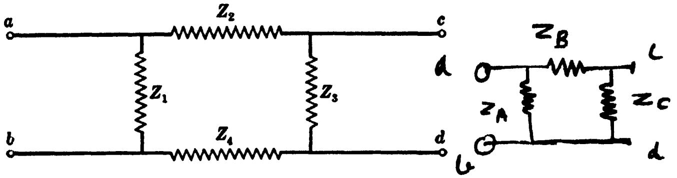


FIG. 2–5. A 4-terminal network.

If one is allowed to make measurements between any pair of terminals the reduction cannot be made. Consider Fig. 2–5. Assume that it is required to reduce this circuit to a π section. Obviously if it can be reduced to a π section it can also be reduced to a T section according to Art. 17. In order to find Z_A, Z_B, and Z_C for the equivalent π section it is necessary to have three defining equations, and for these use Z_{ab}, Z_{ac}, and Z_{cd}. Now suppose the problem is completed and Z_A, Z_B, and Z_C are found to have some definite values. Obviously, in the π section, $Z_{bd} = 0$, and just as obviously, in Fig. 2–5, $Z_{bd} \neq 0$. Thus if it is necessary to have an exact equivalence between the two circuits the problem is impossible. Usually, however, in work of the kind treated here only an equivalence between Z_{ab} and Z_{cd} is necessary. Thus let only impedances involving the input and output terminals be employed.

In order to obtain three equations use

$$Z_{ab} \text{ with } cd \text{ open-circuited} = Z_{abo}$$

$$Z_{ab} \text{ with } cd \text{ short-circuited} = Z_{abs}$$

$$Z_{cd} \text{ with } ab \text{ open-circuited} = Z_{cdo}$$

and

$$Z_{cd} \text{ with } ab \text{ short-circuited} = Z_{cds}$$

Thus there are really four equations for determining the three impedances Z_A, Z_B, and Z_C.

$$\text{For } Z_{abo}\text{: } \frac{Z_1(Z_2 + Z_3 + Z_4)}{Z_1 + Z_2 + Z_3 + Z_4} = \frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C}$$

$$\text{For } Z_{abs}\text{: } \frac{Z_1(Z_2 + Z_4)}{Z_1 + Z_2 + Z_4} = \frac{Z_A Z_B}{Z_A + Z_B}$$

$$\text{For } Z_{cdo}\text{: } \frac{Z_3(Z_1 + Z_2 + Z_4)}{Z_1 + Z_2 + Z_3 + Z_4} = \frac{Z_C(Z_B + Z_A)}{Z_A + Z_B + Z_C}$$

It is to be noted in the above equations that Z_2 and Z_4 always appear together as a sum and nowhere separately. Thus we can write Z' for $(Z_2 + Z_4)$ and rewrite the equations:

$$\frac{Z_1(Z_3 + Z')}{Z_1 + Z_3 + Z'} = \frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C} \tag{2-22}$$

$$\frac{Z_1 Z'}{Z_1 + Z'} = \frac{Z_A Z_B}{Z_A + Z_B} \tag{2-23}$$

$$\frac{Z_3(Z_1 + Z')}{Z_1 + Z_3 + Z'} = \frac{Z_C(Z_B + Z_A)}{Z_A + Z_B + Z_C} \tag{2-24}$$

It is easily seen that all three of these equations would be satisfied if

$$\left.\begin{aligned} Z_A &= Z_1 \\ Z_B &= Z' = Z_2 + Z_4 \\ Z_C &= Z_3 \end{aligned}\right\} \tag{2-25}$$

Thus, if the input and output terminals only are to be used, an equivalent π section can be made to represent the 4-terminal circuit of Fig. 2–5. This is of importance in transmission lines and was assumed in Art. 14 where the impedance of the lower wire was placed in the upper wire.

Another 4-terminal network of importance is the lattice network of Fig. 2–6. Let it be required to find an equivalent T section to represent

the lattice. Using the open- and short-circuit impedances again

$$Z_{abo} = \frac{(Z_A + Z_C)(Z_B + Z_D)}{Z_A + Z_B + Z_C + Z_D} \quad (= Z_1 + Z_3) \qquad [2\text{–}26]$$

$$Z_{abs} = \frac{Z_A Z_B}{Z_A + Z_B} + \frac{Z_C Z_D}{Z_C + Z_D} \quad \left(= Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}\right) \qquad [2\text{–}27]$$

$$Z_{cdo} = \frac{(Z_A + Z_B)(Z_C + Z_D)}{Z_A + Z_B + Z_C + Z_D} \quad (= Z_2 + Z_3) \qquad [2\text{–}28]$$

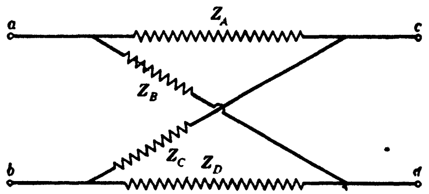


FIG. 2–6. Lattice network.

Substitute Z_{abo}, Z_{abs}, and Z_{cdo} from the above into equation 2–17. After considerable algebraic manipulation there is obtained

$$Z_3 = \frac{\pm(Z_A Z_D - Z_B Z_C)}{(Z_A + Z_B + Z_C + Z_D)}$$

Note that, in Fig. 2–6, if $Z_D = 0$ the circuit reduces to a π section and equation 2–6 applies with the appropriate change in notation. Thus the negative sign in the above equation is to be used to obtain the equivalence for this special case, and the equation is

$$Z_3 = \frac{Z_B Z_C - Z_A Z_D}{Z_A + Z_B + Z_C + Z_D} \qquad [2\text{–}29]$$

Equation 2–29 can be substituted into equation 2–18 (or 2–26) to obtain

$$\begin{aligned} Z_1 &= \frac{(Z_A + Z_C)(Z_B + Z_D) - (Z_B Z_C - Z_A Z_D)}{Z_A + Z_B + Z_C + Z_D} \\ &\quad \frac{Z_A Z_B + Z_A Z_D + Z_C Z_B + Z_C Z_D - Z_B Z_C + Z_A Z_D}{Z_A + Z_B + Z_C + Z_D} \\ &\quad \frac{Z_A Z_B + 2 Z_A Z_D + Z_C Z_D}{Z_A + Z_B + Z_C + Z_D} \end{aligned} \qquad [2\text{–}30]$$

Similarly from equation 2–19 (or 2–28) can be obtained

$$Z_2 = \frac{Z_A Z_C + 2 Z_A Z_D + Z_B Z_D}{Z_A + Z_B + Z_C + Z_D} \qquad [2\text{–}31]$$

Summary. In summarizing the foregoing articles it should be noted that

(*a*) Any 3-terminal network can be reduced to an equivalent T or π section having only three impedances.

(*b*) These impedances may or may not be readily simulated in the laboratory, as, for instance, when one or more of them are composed of negative resistances. However, as far as calculations are concerned they are perfectly satisfactory.

(*c*) Any 4-terminal network used for transmission which has two input terminals and two output terminals has an equivalent T or π section which may be determined by making short- and open-circuit measurements at the input and output terminals.

(*d*) Any circuit having four terminals to be used in this manner can also be reduced to an equivalent T or π section.

PROBLEMS

2-1. If the impedances in Fig. 2-3*a* are all pure resistances (ohms) as follows, find the equivalent T section:

$Z_a = 30$	$Z_f = 10$
$Z_b = 50$	$Z_g = 25$
$Z_c = 20$	$Z_h = 40$
$Z_d = 2$	$Z_i = 60$
$Z_e = 15$	$Z_j = 100$

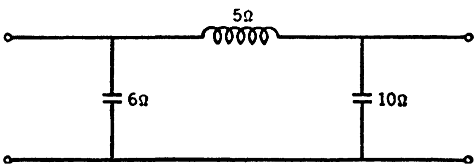


FIG. 2-7. For use in connection with Prob. 2-2.

2-2. Transform the π section of Fig. 2-7 into an equivalent T section.

2-3. Transform the network of Fig. 2-8 into an equivalent T section.

$Z_1 = 50$ ohms resistance	$Z_5 = 10 + j10$ ohms
$Z_2 = j30$ ohms inductance	$Z_6 = 20 - j30$ ohms
$Z_3 = -j20$ ohms capacitance	$Z_7 = 12 + j10$ ohms
$Z_4 = 60$ ohms	

2-4. Transform the resultant T section of Prob. 2-3 into an equivalent π section.

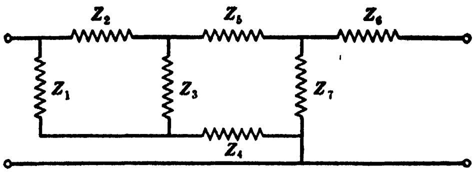


FIG. 2-8. For use in connection with Prob. 2-3.

2-5. Find the equivalent T section of the length of line shown in Fig. 2-9. $f =$ 796 cycles/sec.

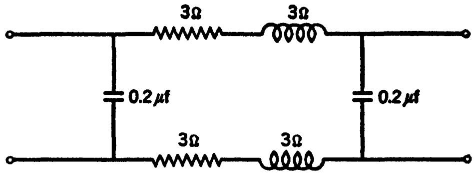


FIG. 2-9. For use in connection with Prob. 2-5.

2-6. Find the open- and short-circuit impedances looking into both ends of the circuit of Fig. 2-10.

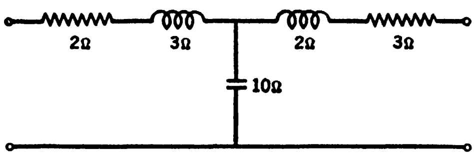


FIG. 2-10. For use in connection with Prob. 2-6.

2-7. For a certain T section

$$Z_{abo} = 6.71\underline{/-63.45^\circ} \text{ ohms}$$

$$Z_{abs} = 7.28\underline{/15.95^\circ} \text{ ohms}$$

$$Z_{cds} = 12.4\underline{/6.17^\circ} \text{ ohms}$$

Find the elements of the T section.

2-8. Show that the T section obtained through use of equations 2-29, 2-30, and 2-31 is equivalent to the lattice network.

2-9. Derive equations for a π section giving Z_A, Z_B, and Z_C in terms of the open- and short-circuit impedance measurements.

CHAPTER III

NETWORK THEOREMS

As mentioned in the previous chapter, network theorems contribute greatly to the ease of handling certain problems involving a considerable number of impedances and generators. It should be noted that only impedances which are linear and bilateral are assumed in these theorems. A linear impedance is one which is independent of the amount of current flowing through it, and a bilateral impedance is one which will conduct electricity equally well in either direction. This eliminates such elements as vacuum tubes, copper oxide rectifier units, etc. The theorems to be taken up in this chapter are: Thévenin's, superposition, reciprocity, compensation, and maximum power transfer. All of these are useful in the material which follows.

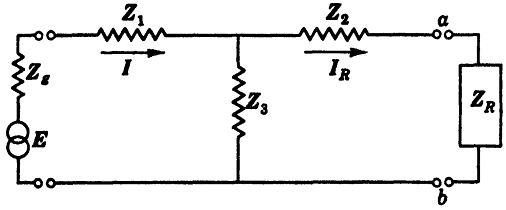


FIG. 3–1. Equivalent T section.

20. Thévenin's Theorem.[1] One form of Thévenin's theorem is: *If a network containing any number of sources of emf has in it two terminals* a *and* b *which are the terminals of a subnetwork, the current through this subnetwork is the same as if it were connected to a source of emf whose open-circuit emf is the voltage across* a *and* b *when the subnetwork is disconnected and whose impedance is the equivalent impedance looking back into the network from the terminals* a *and* b *with all sources of emf replaced by their internal impedances.*

As an illustration of the use of the theorem assume that the original network between the source of emf and the load impedance is replaced by its equivalent T section, as shown in Fig. 3–1, and that Z_1, Z_2, and

[1] For a generalized proof of Thévenin's theorem see *Basic Electrical Engineering* by G. F. Corcoran, p. 149, John Wiley & Sons, New York, 1949.

Z_3 have been determined by means of equations 2–17, 2–18, and 2–19. Then the current I_R is found from the relation,

$$I = \frac{E}{Z_g + Z_1 + \dfrac{Z_3(Z_2 + Z_R)}{Z_2 + Z_3 + Z_R}}$$

Since

$$\frac{I_R}{I - I_R} = \frac{Z_3}{Z_2 + Z_R}$$

it follows that

$$I_R = \frac{Z_3}{Z_2 + Z_3 + Z_R} \cdot \frac{E}{Z_g + Z_1 + \dfrac{Z_3(Z_2 + Z_R)}{Z_2 + Z_3 + Z_R}}$$

$$= \frac{EZ_3}{Z_2(Z_g + Z_1 + Z_3) + Z_3(Z_g + Z_1) + Z_R(Z_g + Z_1 + Z_3)}$$

$$= \frac{\dfrac{EZ_3}{Z_g + Z_1 + Z_3}}{Z_2 + \dfrac{Z_3(Z_g + Z_1)}{Z_g + Z_1 + Z_3} + Z_R} \qquad [3\text{–}1]$$

The voltage across a–b with Z_R disconnected is

$$E_{ab} = \frac{EZ_3}{Z_g + Z_1 + Z_3} \qquad [3\text{–}2]$$

The impedance Z_{ab}, measured back into the terminals a–b with Z_R out, is

$$Z_{ab} = Z_2 + \frac{Z_3(Z_g + Z_1)}{Z_g + Z_1 + Z_3} \qquad [3\text{–}3]$$

Reference to equation 3–1 shows that the current I_R is the same as that obtained from a circuit such as is shown in Fig. 3–2 where E_{ab} as given by equation 3–2 is the source of emf and Z_{ab} as given by equation 3–3 is the equivalent internal impedance of the source, in series with Z_R. Thus the theorem is verified for the network of Fig. 3–1.

21. Superposition Theorem. *If a network has two or more sources of emf in it, the current through any component impedance is the sum of the currents obtained by considering the emf's one at a time, each of the other sources of emf being replaced during this procedure by its internal impedance.*

The truth of this theorem follows from a consideration of the linearity of the impedances because, if the fact that current is already flowing in the impedances cannot change the value of the impedances, when an additional emf is added to the network at any point the current set up by it will not be affected by the current already flowing nor will it affect

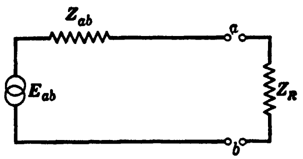


FIG 3-2 Illustration of Thévenin's theorem

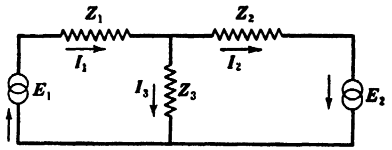


FIG 3-3 Illustration of superposition theorem

the current already flowing To illustrate the superposition theorem by a simple example, consider the network of Fig 3-3 Let it be required to find the current I_3 First it will be found taking into consideration both sources of emf simultaneously

$$E_1 = Z_1 I_1 + Z_3 I_3 = I_1 Z_1 + I_2 0 + I_3 Z_3$$
$$E_2 = Z_2 I_2 - Z_3 I_3 = I_1 0 + I_2 Z_2 - I_3 Z_3$$
$$0 = I_1 - I_2 - I_3$$

Thus

$$\Delta = \begin{vmatrix} Z_1 & 0 & Z_3 \\ 0 & Z_2 & -Z_3 \\ 1 & -1 & -1 \end{vmatrix} = -Z_1Z_2 - Z_2Z_3 - Z_3Z_1$$

and

$$I_3 = \frac{\begin{vmatrix} Z_1 & 0 & E_1 \\ 0 & Z_2 & E_2 \\ 1 & -1 & 0 \end{vmatrix}}{\Delta} = \frac{E_2Z_1 - E_1Z_2}{\Delta}$$

$$= \frac{E_1Z_2 - E_2Z_1}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} \qquad [3\text{-}4]$$

Now consider each source of emf separately Use single primes for currents due to E_1 and double primes for those due to E_2.

$$I_1' = \frac{E_1}{Z_1 + \dfrac{Z_2Z_3}{Z_2 + Z_3}}$$

$$I_3' = \frac{Z_2}{Z_2 + Z_3} \cdot \frac{E_1}{Z_1 + \dfrac{Z_2 Z_3}{Z_2 + Z_3}}$$

$$= \frac{E_1 Z_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}$$

Similarly

$$I_3'' = \frac{E_2 Z_1}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}$$

Now I_3' and I_3'' both flow through Z_3 but in opposite directions, as indicated by the arrow directions of E_1 and E_2. Thus the total current through Z_3 is

$$I_3 = I_3' - I_3''$$

which gives the same result as equation 3–4.

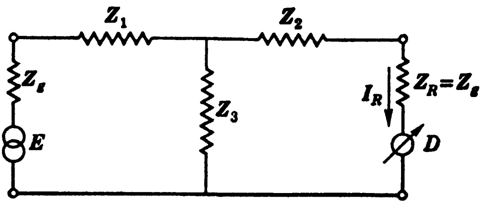


FIG. 3–4. Illustration of reciprocity theorem.

22. Reciprocity Theorem. *If a single source of emf* **E** *produces a current* $\mathbf{I_R}$ *in any given conductor of a network, the same current* $\mathbf{I_R}$ *will be produced at the original location of the source of emf if* **E** *and the detector, reading the current, are interchanged, provided the internal impedance of the source of emf and of the detector are identical.*

As an illustration assume that in Fig. 3–4 the T section represents the network between the source of emf and detector D. The current I_R in D has been shown in Art. 20 to be (when $Z_R = Z_g$)

$$I_R = \frac{E Z_3}{Z_2(Z_g + Z_1 + Z_3) + Z_3(Z_g + Z_1) + Z_g(Z_g + Z_1 + Z_3)}$$

which may be written in the form,

$$I_R = \frac{E Z_3}{(Z_1 + Z_g)(Z_2 + Z_g) + Z_3(Z_1 + Z_g) + Z_3(Z_2 + Z_g)} \qquad [3\text{–}5]$$

Now, if E and D are interchanged, the effect is the same as an inter-

change of $(Z_1 + Z_g)$ and $(Z_2 + Z_g)$. However, it is seen that an interchange of these two impedances in equation 3–5 does not alter the equation. Hence the current I_R maintains the same value on interchange of the emf E and detector D.

23. Compensation Theorem. *In any network containing emf's any impedance may be replaced by a source of emf of zero internal impedance whose emf is equal to the potential difference* (**IZ**) *in magnitude and phase across the impedance.*

It is to be noted that this replacement does not involve any change of current either in direction or in amount anywhere in the network.

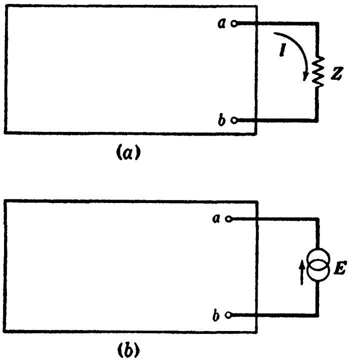


Fig. 3–5. Illustration of compensation theorem.

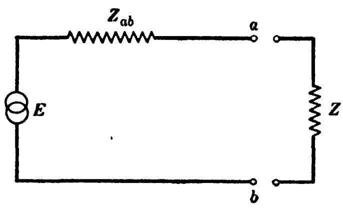


Fig. 3–6. Illustration of maximum power transfer theorem.

Thus this theorem will apply even though the impedances may be nonlinear and unilateral. The truth of the theorem may be seen by referring to Fig. 3–5. Here the impedance in question, Z, has been isolated. A current I in flowing through it produces at the terminals a–b a voltage drop IZ such that a is positive with respect to b. Now as far as the network in Fig. 3–5a, exclusive of Z, is concerned it sees at the terminals a–b nothing but this potential difference, $V_{ab} = IZ$. The amount of current carried by Z is determined by this potential difference V_{ab} only. Thus to the network it is a matter of indifference whether between a and b there is an impedance Z or an emf E, Fig. 3–5b, whose value is IZ and which is connected so that a is the positive terminal of the source of emf for the instant illustrated.

24. Maximum Power Transfer Theorem. *If a source or sources of emf which feed through a network are to supply maximum power to an impedance* **Z** *at the terminals* a–b, **Z** *must be as follows for two different cases:*

(1) *If the angle of* **Z** *is unrestricted,* **Z** *must be the conjugate of the impedance* $\mathbf{Z}_{ab}$ *looking back into the network at the terminals* a–b.

(2) *If the angle of* Z *is fixed, then the absolute values* Z *and* Z_{ab} *must be equal.*

According to Thévenin's theorem the source of emf and network may be represented by the circuit of Fig. 3-6. The current delivered by this network into Z will be

$$I = \frac{E}{Z_{ab} + Z}$$

If $Z = R + jX$, then the power delivered to Z will be

$$P = RI^2 = \frac{E^2R}{|Z_{ab} + Z|^2} \tag{3-6}$$

Let $Z_{ab} = R' + jX'$. Then

$$P = \frac{E^2R}{|R + R' + j(X + X')|^2} = \frac{E^2R}{(R + R')^2 + (X + X')^2} \tag{3-7}$$

It can be seen by inspection that for arbitrary R and R' the maximum power will be obtained when $X = -X'$. Then

$$P_{\text{max}} = \frac{E^2R}{(R + R')^2}$$

Differentiate this expression with respect to R and set the derivative equal to zero in order to determine the value of R for maximum power.

$$\frac{dP_{\text{max}}}{dR} = \frac{(R + R')^2E^2 - 2E^2R(R + R')}{(R + R')^4} = 0$$

Therefore

$$R + R' - 2R = 0$$

and

$$R = R'$$

Thus for maximum power $Z = R' - jX'$ or the conjugate of Z_{ab}.

If the angle of Z is restricted then let

$$Z = Z\underline{/\theta}$$
$$R = Z\cos\theta, \quad \text{and} \quad X = Z\sin\theta$$

Equation 3-7 becomes

$$P = \frac{E^2Z\cos\theta}{|R' + Z\cos\theta + j(X' + Z\sin\theta)|^2}$$

$$= \frac{E^2Z\cos\theta}{(R' + Z\cos\theta)^2 + (X' + Z\sin\theta)^2} \tag{3-8}$$

Differentiate equation 3–8 with respect to Z and set the derivative equal to zero. Then

$$\frac{dP}{dZ} = \frac{[(R' + Z\cos\theta)^2 + (X' + Z\sin\theta)^2]E^2\cos\theta - E^2Z\cos\theta[2(R' + Z\cos\theta)\cos\theta + 2(X' + Z\sin\theta)\sin\theta]}{\text{DENOMINATOR}}$$
$$= 0$$

from which

$$(R' + Z\cos\theta)^2 + (X' + Z\sin\theta)^2$$
$$- 2Z(R'\cos\theta + Z\cos^2\theta) - 2Z(X'\sin\theta + Z\sin^2\theta) = 0$$
$$R'^2 + X'^2 - Z^2\cos^2\theta - Z^2\sin^2\theta = 0$$

or

$$R'^2 + X'^2 = R^2 + X^2$$

Hence the absolute value of Z must be equal to the absolute value of Z_{ab}.

PROBLEMS

3–1. Reduce the network of Fig. 3–7a to the form in Fig. 3–7b.

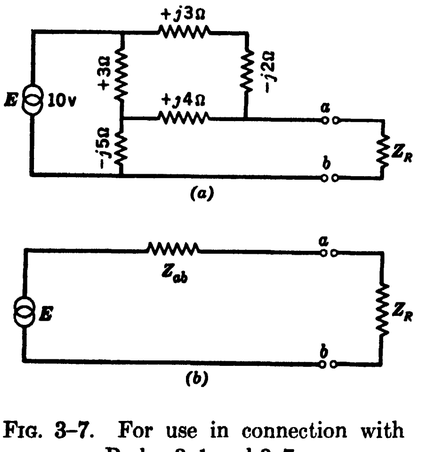


FIG. 3–7. For use in connection with Probs. 3–1 and 3–7.

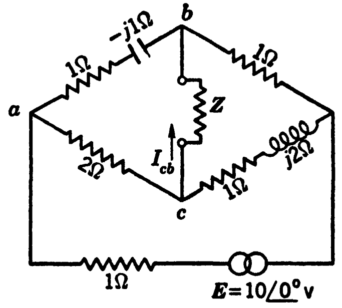


FIG. 3–8. For use in connection with Prob. 3–2.

3–2. A bridge network is shown in Fig. 3–8. Find I_{cb} for

(a) $Z = -j0.434$ ohm,

(b) $Z = 0.32 - j0.434$ ohm.

3–3. In the network shown in Fig. 3–9 a 1-kilohm resistor is to be inserted across the break x-x'. The output voltage of the network is to be taken as the voltage drop across the 1-kilohm resistor. In the final adjustment of the network a maximum output voltage is desired per unit of input voltage E, and it is permissible to insert in series with the output resistor that value of reactance which will maximize

the output voltage. Find the reactance which if placed in series with the output resistor will produce maximum output voltage. Note: Employ Thévenin's theorem.

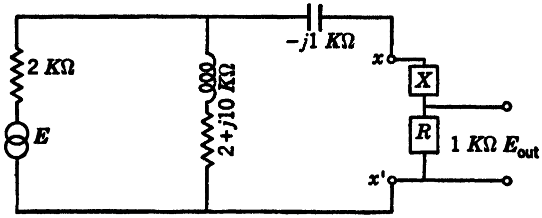


FIG. 3–9. For use in connection with Prob. 3–3.

3–4. Given the network shown in Fig. 3–10, determine I_a in complex rectangular form. Hint: Take advantage of the symmetry by making Thévenin-theorem breaks at the points of symmetry.

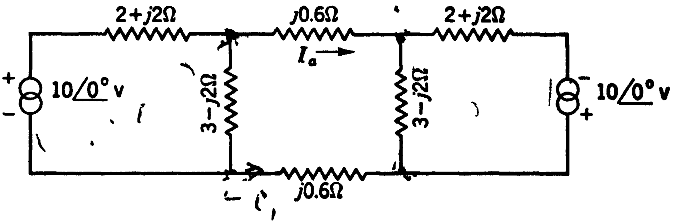


FIG. 3–10. For use in connection with Prob. 3–4.

3–5. A network is shown in Fig. 3–11. Find the current in the 3-ohm resistor by the method of superposition.

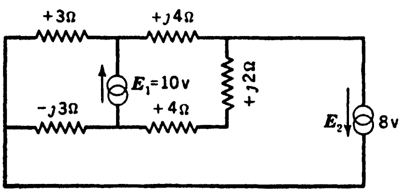


FIG. 3–11. For use in connection with Prob. 3–5.

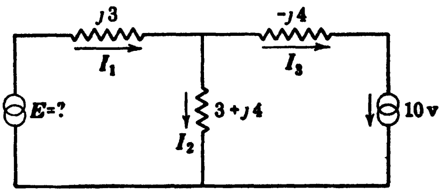


FIG. 3–12. For use in connection with Prob. 3–6.

3–6. Find the value of E in the circuit of Fig. 3–12 by using the method of superposition. $I_1 = 4.88\underline{/57.62^\circ}$ and $I_2 = 4.09\underline{/-58.20^\circ}$ amperes.

3–7. If Z_R in Fig. 3–7 is a 3-ohm resistor, find the emf which can be substituted for Z_R.

3–8. A network can be represented as in Fig. 3–13. Find the value of Z_R which will result in the maximum transfer of power.

3-9. In Prob. 3-8, if Z_R is restricted to an impedance with an angle of 30°, what must be its value for maximum transfer of power?

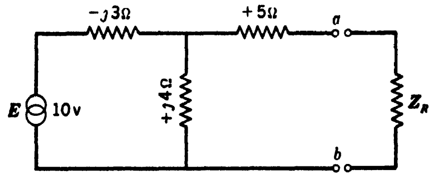


FIG. 3-13. For use in connection with Prob. 3-8.

3-10. Compare the actual power transferred in Prob. 3-8 with that of Prob. 3-9.

3-11. In Prob. 3-8 if Z_R is equal to the impedance measured back into the circuit at a-b find the power transferred.

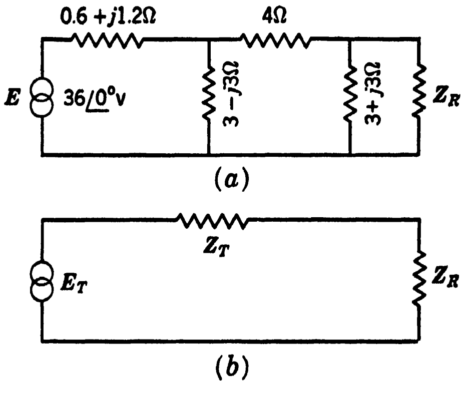


FIG. 3-14. For use in connection with Prob. 3-12.

3-12. Reduce the network of Fig. 3-14a to the form of Fig. 3-14b. Specify clearly

(a) E_T in complex polar form,
(b) Z_T in complex rectangular form,
(c) Z_R in complex rectangular form for maximum transfer of power.

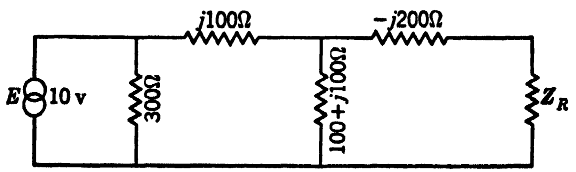


FIG. 3-15. For use in connection with Prob. 3-13.

3-13. Reduce the circuit of Fig. 3-15 to its simplest form by the use of Thévenin's theorem, and find Z_R so that the power delivered to it will be a maximum.

CHAPTER IV

THE LINE COMPOSED OF FINITE SECTIONS

In the previous chapters it has been established that a communication line can be considered as composed of a large number of T (or π) sections in tandem. The treatment of such a line leads to certain basic ideas which are useful in material to follow, although the treatment, as applied to lines in this chapter, is only approximate. This chapter introduces the ideas of *characteristic impedance*, *propagation constant*, and *distortion* and leads to the consideration of certain simple transmission lines.

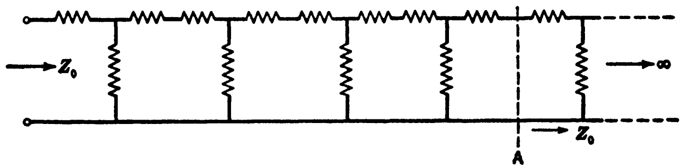


FIG. 4-1. Infinite line constructed of T sections.

25. Characteristic Impedance. A line which is composed of two parallel wires of infinite length has distributed along its entire length, resistance, inductance, capacitance, and leakage conductance. If the line were divided into a great many small sections there would be associated with each of these sections a small L, C, R, and G. The true situation, which is approached by making these sections infinitesimally small, is treated later. However, for the time being, it will be assumed that the sections are of finite length and that each section may be considered as a T section.

The condition shown in Fig. 4-1 exists for a line reaching to infinity in one direction. The input impedance to this line, Z_0, obviously has some definite value. Suppose that at some distance along the line, as at A, the line is cut and the first part is taken away. Then, clearly, the impedance Z_0' looking into what is left (still an infinite length of line) is still equal to Z_0 because conditions have not been changed. This is true whether one section or a hundred sections are cut off at the end of the line, and furthermore there is no restriction on the length of the sections.

For the immediate purpose consider one section only, at the beginning

of the line. Then there exists the condition that the section itself has a certain input impedance, Z_0, and at its output it also works into an impedance of Z_0. This situation is as shown in Fig. 4-2.

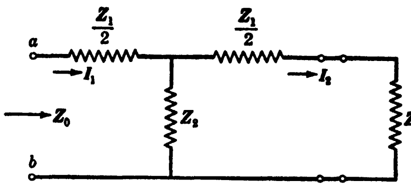


FIG. 4-2. T section terminated in Z_0.

Here it must be noted that the notation has been changed because of the essential equality of the Z_1 and Z_2 as used in the previous chapter. The relation between the old and the new notation, which is shown below, is specifically mentioned because many books on this subject make use of both systems.

OLD SYSTEM	NEW SYSTEM
Z_1	$Z_1/2$
Z_2	$Z_1/2$
Z_3	Z_2

This is certain to cause confusion unless the student is very careful always to distinguish between the two systems. The new system is such that the following holds:

Z_1 is the total series impedance per section,

Z_2 is the total shunt impedance per section.

Referring again to Fig. 4-2, it is seen that there is no possibility of determining by impedance measurements at a–b whether the line is actually infinite toward the right or is terminated, after even one section, by the impedance Z_0. This Z_0 is called *characteristic impedance* or *iterative impedance.* The term *characteristic impedance* will be used in this text. If the section is not uniform, then Z_0 is called the *iterative* impedance only and the name *characteristic* impedance does not apply. The characteristic impedance is the impedance which, if used for a terminating impedance, causes the line to act as an infinite line. Since it is an infinite line a wave started at the sending end will continue to travel out to infinity. Thus a wave sent out on such a line will travel only outward and will never be reflected, just as a water wave in a canal of infinite length will never be reflected because it never meets the end or any discontinuity. In other words, a line terminated in Z_0 will not cause reflections. This will be taken up later in more detail.

It is of interest to determine Z_0 in terms of Z_1 and Z_2. This can be

done using Fig. 4–2 and the following development:

$$Z_0 = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_0\right)}{\frac{Z_1}{2} + Z_2 + Z_0}$$

$$\frac{Z_0Z_1}{2} + Z_0Z_2 + Z_0^2 = \frac{Z_1^2}{4} + \frac{Z_1Z_2}{2} + \frac{Z_1Z_0}{2} + \frac{Z_1Z_2}{2} + Z_0Z_2$$

or

$$Z_0^2 = Z_1Z_2 + \frac{Z_1^2}{4}$$

Thus

$$Z_{0T} = \pm\sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} \qquad [4\text{–}1]$$

where the subscript $0T$ is used to refer to a T section.

It is also useful to have an expression for Z_0 in terms of the open- and short-circuit impedances. First the values of Z_{abo} and Z_{abs} will be calculated and then these will be combined to give equation 4–1.

$$Z_{abo} = \frac{Z_1}{2} + Z_2 \qquad [4\text{–}2]$$

$$Z_{abs} = \frac{Z_1}{2} + \frac{\frac{Z_1Z_2}{2}}{\frac{Z_1}{2} + Z_2} = \frac{\frac{Z_1^2}{4} + Z_1Z_2}{\frac{Z_1}{2} + Z_2} \qquad [4\text{–}3]$$

If equations 4–2 and 4–3 are multiplied together there results

$$Z_{abo}Z_{abs} = \frac{Z_1^2}{4} + Z_1Z_2$$

or on referring to equation 4–1 it is seen that

$$Z_{0T} = \pm\sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} = \pm\sqrt{Z_{abo}Z_{abs}} \qquad [4\text{–}4]$$

Let it be required to find the characteristic impedance of a π section making use of the elements in the T section of Fig. 4–2. This π section and its terminating impedance Z_0 will be as shown in Fig. 4–3. It has been arranged so that the total series impedance and shunt impedance for the section remain the same as for the T section.

Using a procedure similar to that preceding equation 4–1 results in the following development:

$$Z_0 = \frac{2Z_2\left[Z_1 + \dfrac{2Z_2Z_0}{2Z_2 + Z_0}\right]}{2Z_2 + Z_1 + \dfrac{2Z_2Z_0}{2Z_2 + Z_0}}$$

$$= \frac{4Z_1Z_2^2 + 2Z_0Z_1Z_2 + 4Z_0Z_2^2}{4Z_2^2 + 2Z_0Z_2 + 2Z_1Z_2 + Z_0Z_1 + 2Z_0Z_2}$$

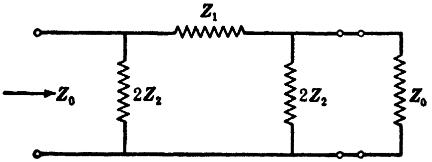


FIG. 4–3. π section terminated in Z_0.

Thus

$$Z_0^2 = \frac{4Z_1Z_2^2}{4Z_2 + Z_1}$$

$$= \frac{Z_1^2Z_2^2}{Z_1Z_2 + \dfrac{Z_1^2}{4}}$$

from which

$$Z_{0\pi} = \frac{Z_1Z_2}{\pm\sqrt{Z_1Z_2 + \dfrac{Z_1^2}{4}}} = \frac{Z_1Z_2}{Z_{0T}} \qquad [4\text{–}5]$$

which can be seen to be the case by reference to equation 4–1. Here the subscript 0π refers to the π section.

When the sections are infinitesimal in length the $Z_1^2/4$ term disappears and then $Z_{0T} \equiv Z_{0\pi}$. (See Chapter V.)

It can also be easily shown that

$$Z_{0\pi} = \pm\sqrt{Z_{abo}Z_{abs}} \qquad [4\text{–}6]$$

where Z_{abo} and Z_{abs} are measured on the π section.

Although both signs before the radical $\sqrt{Z_1Z_2 + \dfrac{Z_1^2}{4}}$ or $\sqrt{Z_{abo}Z_{abs}}$ are of significance in general theory, only the positive radical is employed in subsequent derivations.

26. Illustrative Example. To find the value of Z_{0T} for a T section having

$$Z_1 = 50 + j125 = 134.6 \underline{/68.2°} \text{ ohms}$$

$$Z_2 = 200 - j100 = 223.6 \underline{/-26.6°} \text{ ohms}$$

$$Z_{0T} = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} \qquad [4\text{-}1]$$

$$Z_1Z_2 = 30{,}100 \underline{/41.6°}$$

$$Z_1^2 = 18{,}120 \underline{/136.4°}$$

Thus

$$Z_{0T} = \sqrt{30{,}100 \underline{/41.6°} + \frac{18{,}120 \underline{/136.4°}}{4}}$$

$$= \sqrt{22{,}510 + j20{,}000 - 3280 + j3125}$$

$$= \sqrt{19{,}230 + j23{,}125}$$

$$= 173.6 \underline{/25.1°} \text{ ohms}$$

Let it be required to find also the value of $Z_{0\pi}$ when Z_1 and Z_2 are rearranged into a π section. Here equation 4–5 applies.

$$Z_{0\pi} = \frac{Z_1Z_2}{Z_{0T}}$$

Accordingly

$$Z_{0\pi} = \frac{30{,}100 \underline{/41.6°}}{173.6 \underline{/25.1°}} = 173.4 \underline{/16.5°} \text{ ohms}$$

27. Propagation Constant. A section such as that illustrated in Fig. 4–2 has been shown to represent a part of a line. If the total number of such parts that make up a line is known and if the effect which each section has on the transmitted signal is known, it is possible to calculate the effect of the entire line on the transmitted signal. To this end it is necessary to determine the effect, on a transmitted signal, caused by the impedances of a single section. In Fig. 4–2 it will be considered that the input current is I_1 and that the current delivered into Z_0, that is, into the remainder of the line, if any, is I_2. The current I_2 can be written in terms of I_1 as follows:

$$I_2 = \frac{Z_2}{\frac{Z_1}{2} + Z_{0T} + Z_2} \cdot I_1$$

or

$$\frac{I_1}{I_2} = \frac{\frac{Z_1}{2} + Z_{0T} + Z_2}{Z_2} = \frac{Z_1}{2Z_2} + \frac{\sqrt{Z_1Z_2 + \frac{Z_1^2}{4}}}{Z_2} + 1$$

$$= 1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2}\left(1 + \frac{Z_1}{4Z_2}\right)} = 1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2} + \frac{Z_1^2}{4Z_2^2}} \qquad [4\text{–}7]^1$$

The ratio I_1/I_2 is complex and may be expressed as $A\epsilon^{j\beta}$. In other words the current in passing through this single section is changed in both magnitude and phase in accordance with equation 4–7.

The change in magnitude is caused by the flow of current through the shunt impedance Z_2, and the change in phase is due to the finite amount of time required for the current to be propagated through the section. The change in phase may be interpreted as a time lag between the signal input and the signal output. The angle β is independent of the phase angle which exists between the current and voltage at a particular point on the line, and the two should not be confused.

As an example, find the complex current ratio of the T section having $Z_1 = 50 + j125$ and $Z_2 = 200 - j100$ when terminated in its characteristic impedance. This ratio is given by equation 4–7:

$$\frac{I_1}{I_2} = 1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2} + \frac{Z_1^2}{4Z_2^2}}$$

$$\frac{Z_1}{Z_2} = 0.603 \underline{/94.8^\circ} = -0.0505 + j0.601$$

$$\frac{Z_1^2}{4Z_2^2} = 0.0907 \underline{/189.6^\circ} = -0.0894 - j0.0151$$

Thus

$$\frac{I_1}{I_2} = 1 - 0.0253 + j0.3005 + \sqrt{-0.0505 + j0.601 - 0.0894 - j0.0151}$$

$$= 1.455 + j0.908$$

$$= 1.716 \underline{/31.95^\circ} = 1.716\epsilon^{j31.95^\circ}$$

whence

$$A = 1.716 \quad \text{and} \quad \beta = 31.95^\circ$$

The propagation constant, written as γ, is *defined* as the logarithm to the base ϵ of this complex current ratio. Hence

$$\gamma = \ln \frac{I_1}{I_2} \quad \text{or} \quad \frac{I_1}{I_2} = \epsilon^\gamma \qquad [4\text{–}8]$$

[1] Equation 4–7 is valid also for the π section. See Prob. 4–10.

Thus

$$\gamma\ (= \alpha + j\beta) = \ln A\epsilon^{j\beta} = \ln\left[1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2} + \frac{Z_1^2}{4Z_2^2}}\right] \qquad [4\text{–}9]^2$$

In this equation, α, the real part of γ, is defined as the *attenuation constant* and β is the *wavelength constant,* or *phase constant.* Equation 4–9 may be written as

$$\gamma = \alpha + j\beta = \ln A + \ln \epsilon^{j\beta} = \ln A + j\beta \qquad [4\text{–}10]$$

where β is in radians, and α, expressed as $\ln A$, is the logarithm of the absolute value of the ratio given by equation 4–7. Note then that α expresses the amount of change in the absolute value of the current and β the change in phase of the current. The unit of α is the *neper,* defined by

$$N = \ln \frac{I_1}{I_2} \qquad [4\text{–}11]$$

Thus 1 neper corresponds to a current ratio of 2.71828.

From the previous example $\frac{I_1}{I_2} = 1.716\ \underline{/31.95°}$. Let it be required to find γ, α, and β for the section. From equation 4–10,

$$\begin{aligned} \gamma = \alpha + j\beta &= \ln A + j\beta \\ &= \ln 1.716 + j31.95° \\ &= 0.540 + j0.558 \end{aligned}$$

it means I_1

$\ln \frac{I_1}{I_2} = .540$

Thus

$$\alpha = 0.540 \text{ neper}$$

$$\beta = 0.558 \text{ radian}$$

The above results mean that in traversing the section the current vector magnitude decreases by 0.540 neper and the vector itself is rotated through 0.558 radian.

28. Attenuation. In the study of psychology and acoustics it is known that the ear considers as equal changes those changes which are in the same ratio. Thus the change from 20 to 30 milliwatts (10-milliwatt change) would seem the same as the change from 200 to 300 milliwatts (100-milliwatt change) because 30/20 = 300/200. These levels are equally spaced on a logarithmic scale. This leads to the logical use of some unit in attenuation which is proportional to the logarithm of the

[2] See also equation 7–17 for an alternative expression for γ.

change ratio. The use of a logarithmic scale leads also to a certain simplification in calculations.

The *attenuation factor* is defined as the ratio of currents I_1/I_2 ($= \epsilon^{\alpha}$). In the previous article the logarithmic method of representation was assumed where there was written $\alpha = \ln I_1/I_2$. Now it is advisable to investigate the procedure for taking care of a number of sections in

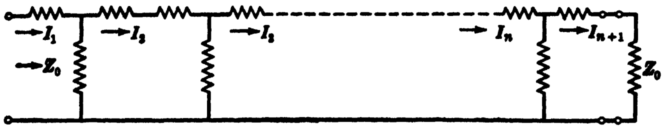


FIG. 4-4. Long line terminated in Z_0.

tandem, each of which has a certain value of α. Consider Fig. 4-4. Over any number of these sections the attenuation factor will be

$$\frac{I_1}{I_{n+1}} = \frac{I_1}{I_2}\,\frac{I_2}{I_3}\,\frac{I_3}{I_4}\,\frac{I_4}{I_5}\,\frac{I_5}{I_6}\,\cdots\,\frac{I_n}{I_{n+1}}$$

The propagation constant will be

$$\begin{aligned}\gamma_t = \alpha_t + j\beta_t &= \ln \frac{I_1}{I_{n+1}} \\ &= (\alpha_1 + \alpha_2 + \alpha_3 + \cdots) + j(\beta_1 + \beta_2 + \beta_3 + \cdots)\end{aligned}$$

where

$$\alpha_t = \ln \frac{I_1}{I_2} + \ln \frac{I_2}{I_3} + \ln \frac{I_3}{I_4} + \cdots = \alpha_1 + \alpha_2 + \alpha_3 + \cdots$$

Thus the total attenuation constant α_t for n sections in tandem is equal to the sum of the attenuation constants of the separate sections, and β_t is the sum of the individual phase shifts of the n sections. If the sections are all alike, the total attenuation in nepers is $N = n\alpha$ and the total phase shift in radians is $n\beta$.

As shown above, the neper corresponds to a current ratio change of 2.71828. This is a rather large change, and accordingly a new unit is defined on the basis of power ratio and logarithms to the base 10. Thus the *bel* is defined:

$$\text{Number of bels} = \log \frac{P_1}{P_2}$$

where P_1 and P_2 are the input and output powers respectively. For a practical unit the decibel which is one tenth of a bel is used. The num-

ber of decibels (db) is given by

$$\text{Number of db} = 10 \log \frac{P_1}{P_2} \quad [4\text{–}12]$$

which is a positive quantity in losses, that is, for $P_1 > P_2$. One decibel thus corresponds to a power ratio of 1.259. If the sending- and receiving-end impedances of a line remain constant at the value Z_0 during a ratio change, and since power is proportional to the current squared, it follows that

$$\text{db} = 10 \log \left(\frac{I_1}{I_2}\right)^2$$

$$= 20 \log \frac{I_1}{I_2} \quad [4\text{–}13]$$

Since Z_0 is independent of the position along the line at which it is measured:

$$E_1 = Z_0 I_1$$

$$E_2 = Z_0 I_2$$

so that the ratio E_1/E_2 is the same as I_1/I_2. Thus

$$\text{db} = 20 \log \frac{E_1}{E_2} \quad [4\text{–}14]$$

One decibel thus corresponds to a current (or voltage) ratio of 1.122.

$$1 \text{ db} \equiv 0.115 \text{ neper} \qquad 1 \text{ neper} \equiv 8.686 \text{ db}$$

In its original definition, the decibel is the unit of power attenuation whereas the neper is a unit of attenuation of current (or voltage). If, however, the network is terminated in its characteristic impedance either decibels or nepers may be employed to specify the attenuation, and it is in this sense that the neper is a unit of attenuation which is 8.686 times larger than the decibel.

29. Illustrative Example. Find the ratio of power input to power output for the T section of the preceding examples and determine the loss in decibels.

The current ratio has been given as 1.716. According to equation 4–13 this corresponds to a number of decibels given by

$$\text{db} = 20 \log 1.716$$

$$= 20 \times 0.234 = 4.68$$

Since the section has been terminated in its characteristic impedance, its input impedance is also Z_{0T} (= $Z_{0T} \underline{/\theta°}$). The input power thus becomes

$$P_1 = I_1^2 Z_{0T} \cos\theta$$

and the output power is

$$P_2 = I_2^2 Z_{0T} \cos\theta$$

The power ratio then becomes

$$\frac{P_1}{P_2} = \frac{I_1^2}{I_2^2} = (1.716)^2 = 2.94$$

Equation 4–12 gives

$$\text{db} = 10 \log 2.94 = 4.68$$

which agrees with the decibel loss calculated from the current ratio. It will be noted, however, that this agreement depends on the equality of the input and output impedances.

30. Line Attenuation and Power Levels. Obviously attenuation is detrimental to the best interests of the communication company as it not only means low transmission efficiency but, what is more important, it introduces distortion due to the fact that α differs as the frequency changes. These facts necessitate a careful consideration of its effects and present the problem of its elimination if possible. As an illustration to bring out a few interesting facts, consider a hypothetical unloaded cable pair 2000 miles long made of No. 19 AWG copper wire which is terminated in its characteristic impedance and which has the following characteristics at 796 cycles per second:

$$\alpha = 0.1118 \text{ neper/mile}$$

$$Z_0 = 524.7 \text{ ohms}$$

The total N for the entire line is $N = 0.1118 \times 2000 = 223.6$ nepers, and the ratio of input to output current $I_s/I_r = \epsilon^{223.6} \doteq 10^{97}$. This means that if one ampere were introduced at the sending end the output current I_r would be $1/10^{97} = 10^{-97}$ ampere. Since there is a flow of 63×10^{17} electrons per second for one ampere, the number of electrons per second received at the receiving end would be

$$63 \times 10^{17} \times 10^{-97} = 63 \times 10^{-80}$$

or one electron would arrive at the receiving end every 500×10^{68} years. This brief summary indicates that attenuation is very detrimental on actual commercial lines.

The power ratio for the above line would be

$$\left(\frac{I_s}{I_r}\right)^2 = (10^{97})^2 = 10^{194}$$

Suppose one microwatt were required at the receiving end; then the input power would be $10^{194} \times 10^{-6} = 10^{188}$ watts, which would be suspected of being more power than is available in the world. Obviously such a line used in this way would not work.

The difficulty is overcome by providing amplifiers, or so-called "repeaters," spaced along the line at reasonable intervals. One repeater at the line center would not be satisfactory because a simple calculation shows that it would receive power at too low a level (down in the noise area) and would have to supply power at such a high level that suitable vacuum tubes are not available nor likely to be for some time. The student is advised to work out this problem for himself to see just what is involved. A commercial repeater may have a gain, as an illustration, of 60 decibels and for this purpose may be considered to deliver a power of 5 watts.

The power ratio corresponding to 60 decibels is given by R in

$$-60 = 10 \log R \quad \text{(See equation 4–12)}$$

$$\log R = -6$$

$$R = 10^{-6}$$

where the negative sign appears before the 60 because there is a gain. The input power $= 5 \times 10^{-6} = 5$ microwatts.

Since the repeater has a gain of 60 decibels, receiving power at 5 microwatts and delivering at 5 watts, it should be placed at points of the line where the power level has dropped to 5 microwatts. The repeater will then raise the power level at this point to 5 watts and the line from this repeater to the next will again reduce the power to 5 microwatts, and so on across the country. Thus each section of the line produces an attenuation of 60 decibels. In this hypothetical case, what should be the distance between repeaters?

The current ratio corresponding to 60 decibels is given by R_c in

$$60 = 20 \log R_c \quad \text{(See equation 4–13)}$$

$$\log R_c = 3$$

$$R_c = 1000 = \frac{I_1}{I_2}$$

$$\text{Nepers, } N = \ln 1000 = 0.1118 \times \text{number of miles}$$

Therefore

$$\text{Number of miles} = \frac{6.91}{0.1118} = 61.8$$

Thus in the 2000 miles of line, 33 repeaters would be necessary to accomplish transmission, and at no time would the power level be either too low or too high.

The above illustration which employs both nepers and decibels should serve as an introduction to the use of these units. It should be kept in mind that the assumption was implicitly made that the line impedance remained constant at Z_0 throughout the calculation. Use was made of this fact when it was stated that the 60 decibels represented both the power gain and the current gain.

From what has been considered previously it is easily seen that, since α is a function of Z_1 and Z_2, and thus of the frequency, the amount of amplifier gain must be different for different frequencies. The difference in attenuation for different frequencies produces a distortion of the received signal which is called *frequency distortion*. Also, since β is a function of frequency, the phase of signals at different frequencies may not be a linear function of the frequency, thus causing a delay or advance of certain parts of the range over other parts. This is called *delay distortion*.

The immediate question which may be asked now is "What can be done to decrease attenuation and to make β the correct function of frequency?" These questions are considered in the next chapter.

PROBLEMS

4-1. Calculate Z_{0T}, α and β at 1592 cycles per second for the T section shown in Fig. 4-5.

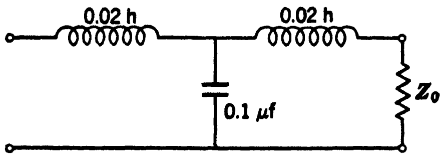


FIG. 4-5. For use in connection with Prob. 4-1.

4-2. (*a*) Calculate $Z_{0\pi}$ at 1592 cycles per second for the π section shown in Fig. 4-6.

(*b*) Calculate the attenuation in decibels and phase shift in degrees at 1592 cycles per second.

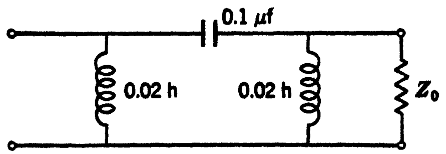


FIG. 4-6. For use in connection with Prob. 4-2.

4-3. A line represented as a T section has a series impedance, $Z_1 = 80 + j\omega(0.050)$ and a shunt impedance, $Z_2 = -j/(\omega 0.620 \times 10^{-6})$ ohms. Find its characteristic impedance at 796 cycles per second.

4-4. Find the characteristic impedance if the impedance elements of Prob. 4-3 are rearranged into a π section.

4-5. Find α and β for the section of line mentioned in Prob. 4-3 if it is terminated in Z_0.

4-6. Assume that a line is composed of 20 T sections such as in Prob. 4-3. Find the total attenuation in decibels, assuming that the line is properly terminated.

4-7. If the line of Prob. 4-6 has a 1-volt generator with an internal impedance of 100 ohms resistance connected at the sending end, find the current at the receiving end and the power received.

4-8. Calculate the efficiency of the line of Prob. 4-7.

4-9. A portion of a line is represented as a π section in which $Z_A = Z_C = -j/(\omega 0.500 \times 10^{-6})$ and $Z_B = 50 + j\omega 0.040$ ohms. Find the characteristic impedance of this section at 1000 cycles per second.

4-10. Prove that equation 4-7 is valid for the π section as shown in Fig. 4-3.

4-11. Find α and β for the π section of Prob. 4-9. The π section is to be terminated in its characteristic impedance.

4-12. Ten T sections obtained through rearrangement of the elements of the π section of Prob. 4-9 are connected in tandem, and the resulting line is terminated in its characteristic impedance. If the input current to the line is 10 milliamperes, find the current and the voltage at each junction between sections.

4-13. (*a*) Determine the maximum length of line that can be used to transmit power under the following conditions:

Sending-end power = 5×10^{-5} watt
Receiving-end power = 10×10^{-5} watt
Line attenuation = 0.10 neper per mile
One repeater station having a gain of 60 db

(*b*) What is the maximum distance from the sending end to the repeater in order that the power input to the repeater be not less than 5×10^{-6} watt?

(*c*) What is the power output of the repeater in part *b*?

4-14. A 20-mile length of coaxial transmission line has a characteristic impedance of $50\angle 0°$ ohms and an attenuation of 2 decibels per mile. Assuming that the line is terminated in its characteristic impedance and that the power input is 0.5 watt, find (*a*) the output power in milliwatts, (*b*) the output current in milliamperes.

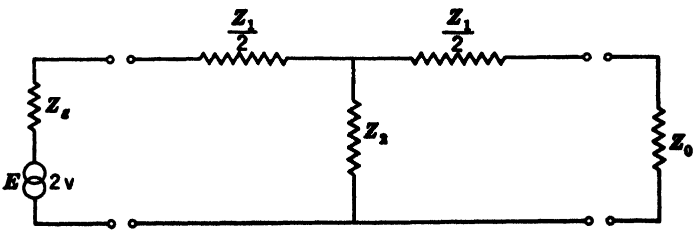


FIG. 4-7. For use in connection with Prob. 4-15.

4-15. A line is made up of 15 sections as shown in Fig. 4-7 and is terminated in its characteristic impedance. $Z_1 = 20\angle 30°$ ohms and $Z_2 = 250\angle 45°$ ohms.

(*a*) Find the input impedance.
(*b*) If $Z_g = 600\angle 0°$ ohms, find the input current and power.
(*c*) Find the total attenuation in decibels.
(*d*) Find the efficiency of the line.

4-16. Assume that the 2000-mile line of Art. 30 has an input power of 5 watts and must deliver 1 microwatt at the receiving end. A repeater is to be placed at the center of the line to make this possible. (*a*) What is the power input to the repeater? (*b*) What is the power output? (*c*) Find the gain of the repeater in decibels. (*d*) Is such an amplifier possible with present-day equipment?

CHAPTER V

THE LINE HAVING UNIFORMLY DISTRIBUTED PARAMETERS

The natures of Z_0, α, and β have been touched upon, having been defined and treated on the basis of finite sections of line. It is now proposed to determine their expressions as used on a line which has uniformly distributed parameters instead of lumped parameters and whose length is either infinite or properly terminated. This problem involves the cutting of the line into incremental lengths and using the methods of the calculus. The present chapter will proceed through that development and will treat a few special lines which are of particular interest. The material leads to the consideration of reflection caused by improper terminations, that is, by terminating impedances different from Z_0.

31. An Infinity of Infinitesimals. Assume that a line is divided into an infinity of infinitesimal lengths, dx, and that one such section of the line is as shown in Fig. 5-1. Each length dx has associated with it an incremental Z_1 and Z_2. A change in previous notation is made as follows:

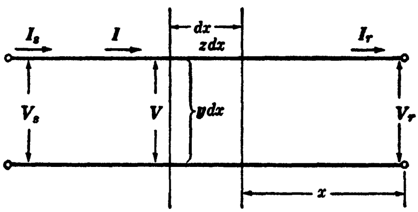


FIG. 5-1. Element of transmission line.

Let z mean Z_1, series impedance per unit length (usually a mile) of line, and y mean $1/Z_2$, or the line shunt admittance per unit length.

Thus for the elemental length dx the series impedance is zdx, and the shunt admittance is ydx. At the input terminals of the element in question the voltage across the line is V, and the current into the element is I. The voltage drop across the element, along the line is

$$dV = Izdx$$

and as the current flows through the element, there is a decrease

$$dI = Vydx$$

The following development is based on these equations.

$$\frac{dV}{dx} = Iz \tag{5-1}$$

$$\frac{dI}{dx} = Vy \tag{5-2}$$

Differentiating equation 5–1 gives

$$\frac{d^2V}{dx^2} = z\frac{dI}{dx}$$

Substituting equation 5–2 into this equation gives

$$\frac{d^2V}{dx^2} = zyV \tag{5-3}$$

Similarly, by first differentiating equation 5–2 and then substituting from equation 5–1 there is obtained

$$\frac{d^2I}{dx^2} = zyI \tag{5-4}$$

The usual procedure in electrical engineering is to assume solutions of these equations and try them out. Based on a study of differential equations, a safe guess for the solution of equation 5–3 is $V = A\epsilon^{ax}$. If this turns out satisfactorily the solution of equation 5–4 is most easily found by substituting for V in equation 5–1. Making the above substitution in equation 5–3

$$\frac{d^2V}{dx^2} = Aa^2\epsilon^{ax} = zyA\epsilon^{ax}$$

From this it is seen that

$$a^2 = zy$$

or

$$a = \pm\sqrt{zy}$$

Thus, since a takes two values, the solution becomes

$$V = A\epsilon^{\sqrt{zy}\,x} + B\epsilon^{-\sqrt{zy}\,x} \tag{5-5}$$

Substituting into equation 5–1, there results

$$I = \frac{1}{z}\frac{dV}{dx}$$

$$= \frac{A\sqrt{zy}}{z}\epsilon^{\sqrt{zy}\,x} - \frac{B\sqrt{zy}}{z}\epsilon^{-\sqrt{zy}\,x}$$

$$= A\sqrt{\frac{y}{z}}\,\epsilon^{\sqrt{zy}\,x} - B\sqrt{\frac{y}{z}}\,\epsilon^{-\sqrt{zy}\,x} \qquad [5\text{–}6]$$

The constants of integration A and B can be evaluated from any known boundary conditions. For boundary conditions use V_r and I_r as receiving-end values when $x = 0$. For this condition then

$$V_r = A + B \qquad \text{(from equation 5–5)}$$

and

$$I_r = A\sqrt{\frac{y}{z}} - B\sqrt{\frac{y}{z}} \qquad \text{(from equation 5–6)}$$

Solving for A and B

$$V_r\sqrt{\frac{y}{z}} = A\sqrt{\frac{y}{z}} + B\sqrt{\frac{y}{z}}$$

$$I_r = A\sqrt{\frac{y}{z}} - B\sqrt{\frac{y}{z}}$$

or adding

$$2A\sqrt{\frac{y}{z}} = V_r\sqrt{\frac{y}{z}} + I_r$$

Therefore

$$A = \frac{V_r}{2} + \sqrt{\frac{z}{y}}\,\frac{I_r}{2} \qquad [5\text{–}7]$$

and

$$B = V_r - A = \frac{V_r}{2} - \sqrt{\frac{z}{y}}\,\frac{I_r}{2} \qquad [5\text{–}8]$$

Equation 5–5 becomes

$$V = \left[\frac{V_r}{2} + \sqrt{\frac{z}{y}}\,\frac{I_r}{2}\right]\epsilon^{\sqrt{zy}\,x} + \left[\frac{V_r}{2} - \sqrt{\frac{z}{y}}\,\frac{I_r}{2}\right]\epsilon^{-\sqrt{zy}\,x}$$

$$V = V_r \left[\frac{\epsilon^{\sqrt{zy}\,x} + \epsilon^{-\sqrt{zy}\,x}}{2} \right] + I_r \sqrt{\frac{z}{y}} \left[\frac{\epsilon^{\sqrt{zy}\,x} - \epsilon^{-\sqrt{zy}\,x}}{2} \right]$$

$$= V_r \cosh \sqrt{zy}\,x + I_r \sqrt{\frac{z}{y}} \sinh \sqrt{zy}\,x \qquad [5\text{–}9]^1$$

In the same manner, equation 5–6 becomes

$$I = \left[\frac{V_r}{2} + \sqrt{\frac{z}{y}}\frac{I_r}{2} \right] \sqrt{\frac{y}{z}}\, \epsilon^{\sqrt{zy}\,x} - \left[\frac{V_r}{2} - \sqrt{\frac{z}{y}}\frac{I_r}{2} \right] \sqrt{\frac{y}{z}}\, \epsilon^{-\sqrt{zy}\,x}$$

$$= V_r \sqrt{\frac{y}{z}} \left[\frac{\epsilon^{\sqrt{zy}\,x} - \epsilon^{-\sqrt{zy}\,x}}{2} \right] + I_r \left[\frac{\epsilon^{\sqrt{zy}\,x} + \epsilon^{-\sqrt{zy}\,x}}{2} \right]$$

$$= I_r \cosh \sqrt{zy}\,x + V_r \sqrt{\frac{y}{z}} \sinh \sqrt{zy}\,x \qquad [5\text{–}10]$$

Equations 5–9 and 5–10 are general equations of a transmission line when the distance is measured from the receiving end. These equations can be written in terms of sending-end voltage and current and with distance measured from the sending end, provided $-x$ is substituted for x in equations 5–9 and 5–10. Since, in the above derivation, positive x direction is measured back along the line from the receiving end (with $x = 0$ at the receiving end), distances along the line (with $x = 0$ at the sending end) are to be regarded as $-x$. Substituting $-x$ for x in these equations gives

$$V = V_s \cosh (-\sqrt{zy}\,x) + I_s \sqrt{\frac{z}{y}} \sinh (-\sqrt{zy}\,x)$$

$$I = I_s \cosh (-\sqrt{zy}\,x) + V_s \sqrt{\frac{y}{z}} \sinh (-\sqrt{zy}\,x)$$

or

$$V = V_s \cosh \sqrt{zy}\,x - I_s \sqrt{\frac{z}{y}} \sinh \sqrt{zy}\,x \qquad [5\text{–}11]$$

$$I = I_s \cosh \sqrt{zy}\,x - V_s \sqrt{\frac{y}{z}} \sinh \sqrt{zy}\,x \qquad [5\text{–}12]$$

since $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$.

32. Determination of Z_0. Let equation 5–11 be used, and impose the condition that the line be infinitely long. Under this condition it

[1] Refer to Appendix III for a discussion of hyperbolic functions.

has been seen that the input impedance is the characteristic impedance Z_0. Thus, if the input impedance can be found, Z_0 is determined. At infinity both V and I are zero. Thus, leaving in the x which $\rightarrow \infty$, we obtain from equation 5–11

$$V_s \cosh \sqrt{zy}\, x = I_s \sqrt{\frac{z}{y}} \sinh \sqrt{zy}\, x \qquad [5\text{–}13]$$

Hence from equation 5–13

$$\frac{V_s}{I_s} = Z_s = \sqrt{\frac{z}{y}} \frac{\sinh \sqrt{zy}\, x}{\cosh \sqrt{zy}\, x} = \sqrt{\frac{z}{y}} \tanh \sqrt{zy}\, x \qquad [5\text{–}14]$$

$$= \sqrt{\frac{z}{y}} = Z_0 \qquad [5\text{–}15]$$

since $\tanh \infty = 1$.

Let us note in passing how this Z_0 compares with that found previously for a finite section of line. In Chapter IV

$$Z_0 = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} \qquad [4\text{–}1]$$

Since $Z_2 = 1/y$, and Z_1 is the present z

$$Z_0 = \sqrt{\frac{z}{y} + \frac{z^2}{4}} \qquad [4\text{–}1a]$$

In this derivation

$$Z_0 = \sqrt{\frac{z}{y}}$$

Th~~us the~~ ~~difference~~ lies in the $z^2/4$ term which is not present in the treatment of the uniform line or when the sections are infinitesimal. (See Appendix IV.)

The Z_0 as given in equation 5–15 for a line with distributed parameters besides being called characteristic impedance is also called *surge* impedance.

33. Illustrative Example. A transmission line, constructed of 104-mil-diameter copper wire with an 18-inch spacing between wires, has the following measured parameters per loop mile (measured C will differ from that calculated by means of equation 1–32 because of the effective relative permittivity of the medium between the conductors differing from unity):

$R = 10.15$ ohms

$L = 3.93$ mh

$C = 0.00797\ \mu$f wire to wire

$G = 0.29\ \mu$ mho

Determine the characteristic impedance Z_0 for this line at 796 cycles per second ($\omega = 5000$) using equation 5–15 for the infinitely long line, and compare with that found by using equation 4–1a.

$$z = R + j\omega L = 10.15 + j5000 \times 0.00393 = 10.15 + j19.65$$
$$= 22.12 \,\angle 62.68^\circ \text{ ohms}$$

$$y = G + j\omega C = (0.29 + j5000 \times 0.00797) \times 10^{-6}$$
$$= (0.29 + j39.85) \times 10^{-6} = 39.85 \times 10^{-6} \,\angle 89.58^\circ \text{ mho}$$

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{22.12 \,\angle 62.68^\circ}{39.85 \times 10^{-6} \,\angle 89.58^\circ}} = 10^3 \sqrt{0.555 \,\angle -26.90^\circ}$$
$$= 745 \,\angle -13.45^\circ \text{ ohms}$$

From equation 4–1a for a section of line one mile long,

$$Z_0 = \sqrt{\frac{z}{y} + \frac{z^2}{4}} = \sqrt{555 \times 10^3 \,\angle -26.90^\circ + \tfrac{1}{4}(22.12 \,\angle 62.68^\circ)^2}$$
$$= \sqrt{554{,}880 \,\angle -26.89^\circ} = 744.9 \,\angle -13.45^\circ \text{ ohms}$$

If the section is 1000 feet long instead of 1 mile, then

$$Z_0 = \sqrt{555 \times 10^3 \,\angle -26.90^\circ + \tfrac{1}{4}(4.19 \,\angle 62.68^\circ)^2}$$
$$= 745 \,\angle -13.45^\circ \text{ ohms}$$

34. Determination of Propagation Constant. The propagation constant represents the change in magnitude and phase in a wave as it passes along the line. Thus, in order to find this constant, it would be helpful to have an equation of V in terms of V_s and x or an equation for I in terms of I_s and x.

Written in terms of exponentials, equation 5–12 is

$$I = I_s \frac{\epsilon^{\sqrt{zy}\,x} + \epsilon^{-\sqrt{zy}\,x}}{2} - \frac{V_s}{Z_0} \frac{\epsilon^{\sqrt{zy}\,x} - \epsilon^{-\sqrt{zy}\,x}}{2} \qquad [5\text{–}16]$$

$$= \left[\frac{I_s - \dfrac{V_s}{Z_0}}{2}\right] \epsilon^{\sqrt{zy}\,x} + \left[\frac{I_s + \dfrac{V_s}{Z_0}}{2}\right] \epsilon^{-\sqrt{zy}\,x} \qquad [5\text{–}17]$$

If the line is terminated in its characteristic impedance, then the input impedance is Z_0, and $V_s/Z_0 = I_s$. The first term of equation 5–17 then drops out, and the equation reduces to

$$I = I_s\epsilon^{-\sqrt{zy}\,x} \qquad I_s\epsilon^{-\gamma x} \qquad [5\text{–}18]$$

Thus at a distance l we have

$$\frac{I_s}{I_l} = \epsilon^{\gamma l} \qquad [5\text{–}19]$$

γl:

or

$$\ln \frac{I_s}{I_l} = \gamma l$$

where γl now plays the same part as the γ of equation 4–8, which defined the propagation constant. Thus γl is the propagation constant for the length under consideration, and γ ($= \sqrt{zy}$) is the propagation constant per unit length, which is taken here to be the mile.

Similarly, equation 5–11 will yield

$$\frac{V_s}{V_l} = \epsilon^{\gamma l} \qquad [5\text{–}21]$$

or

$$\ln \frac{V_s}{V_l} = \gamma l \qquad [5\text{–}22]$$

The constant γ was given in Chapter IV as a more or less complicated function of Z_1 and Z_2. The great simplification in its present expression is to be noted. In order to determine α and β, the real and imaginary parts of γ as previously defined, it is only necessary to evaluate $\sqrt{zy}$ as a complex quantity and take the real and imaginary parts.

In equation 5–18 (and 5–21 also) γ may be complex. Thus

$$\begin{aligned} I_x &= I_s \epsilon^{-(\alpha + j\beta)x} \\ &= I_s \epsilon^{-\alpha x} \epsilon^{-j\beta x} \end{aligned}$$

$$\frac{I_x}{I_s} = \epsilon^{-\alpha x}(\cos \beta x - j \sin \beta x) = \epsilon^{-N} \underline{/-\beta x} \qquad [5\text{–}23]$$

This shows that as one proceeds along the line the absolute value I_x/I_s decreases according to the constant α, and the vector I_x/I_s rotates clockwise at a rate fixed by β. This is repeating again what has already been pointed out concerning α and β.

In equation 5–23, β occurs as circular radians per unit length. Since α occurs in the equation in a manner similar to β, it might be expected that α also should be expressible in something analogous to radians. Equation 5–23 can be written as

$$\frac{I_x}{I_s} = (\cosh \alpha x - \sinh \alpha x)(\cos \beta x - j \sin \beta x) \qquad [5\text{–}24]$$

wherein the complex ratio I_x/I_s is expressed in terms of hyperbolic and circular functions. (See Appendix III.) The β occurs only in the circular functions and, as before mentioned, is in radians per mile. The attenuation α occurs only in the hyperbolic functions and by analogy may be said to be in hyperbolic radians per mile. Thus it is seen that the *neper* corresponds to a unit called the "hyperbolic radian per mile," and thus there is an analogy between α and β which is useful to keep in mind although the concept of a hyperbolic radian per mile is difficult to visualize.

35. Illustrative Example. Consider the line of Art. 33 to be 200 miles in length, and determine the relation between the vector I_x/I_s and distance x.

$$\gamma = \sqrt{zy} = \sqrt{22.12\underline{/62.68^\circ} \times 39.85 \times 10^{-6}\underline{/89.58^\circ}}$$

$$= 0.0297\underline{/76.13^\circ} = 0.00712 + j0.0288$$

$$\alpha = 0.00712 \text{ neper/mile}$$

$$\beta = 0.0288 \text{ radian/mile}$$

$$\frac{I_x}{I_s} = \epsilon^{-0.00712x}\epsilon^{-j0.0288x} = \epsilon^{-0.00712x}\underline{/-0.0288x}$$

For $x = 50$ miles

$$\frac{I_x}{I_s} = \epsilon^{-0.356}\underline{/-1.44 \text{ radians}} = 0.70\underline{/-82.5^\circ}$$

x	$\epsilon^{-\alpha x}$	βx°
0	1	0
25	0.836	−41.3
50	0.700	−82.5
75	0.586	−123.8
100	0.490	−165.0
125	0.411	−206.3
150	0.344	−247.5
175	0.288	−289.0
200	0.241	−330.0

The variation of I_x/I_s with distance is a logarithmic spiral as shown in Fig. 5–2. As attenuation decreases, the locus of I_x/I_s gradually evolves into a circle.

36. Wavelength and Velocity of Propagation. From equation 5–23 and the immediately preceding material it is seen that β represents the rotation of the V or I vector with respect to V_s or I_s in radians per unit length. Thus, in this case, β is given in radians per mile. A rotation of 360° subtends a complete wave or effects a change from a positive

maximum at $x = x_1$ to the next positive maximum at $x = x_1 + \lambda$. This λ is known as the wavelength, and it follows that

$$\beta\lambda = 2\pi \qquad [5\text{–}25]$$

or

$$\lambda = \frac{2\pi}{\beta} \qquad [5\text{–}26]$$

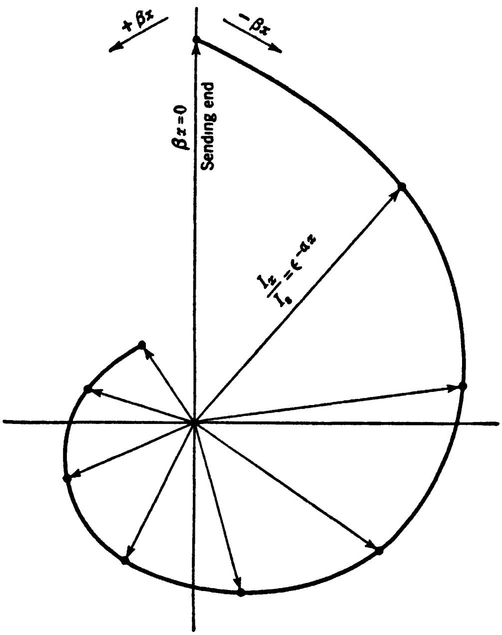


FIG. 5–2. Illustration of attenuation and phase shift.

Let v be the velocity of propagation or phase velocity in miles per second. In one second the vector will rotate through a number of radians equal to $v\beta$ which is then equal to the number of radians per second. However, the number of radians per second is the frequency multiplied by 2π. Therefore

$$v\beta = 2\pi f \qquad [5\text{–}27]$$

or

$$v = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} \qquad [5\text{–}28]$$

Thus, if β were known, both the wavelength and the velocity could be found from equations 5–26 and 5–28. It will be shown later that

the velocity of propagation may lie between the limits of about 10,000 miles per second and the velocity of light, 186,285 miles per second, depending on the line constants. It thus becomes helpful to have equations giving β (and α) explicitly in terms of the fundamental constants.

37. Illustrative Example. To determine the wavelength of the line considered in Art. 33 and the velocity of propagation of the 796 cycles-per-second wave on this line. The value of β was determined in Art. 35 and found to be 0.0288 radian per mile. Hence

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.0288} = 218 \text{ miles}$$

and

$$v = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} = \frac{5000}{0.0288} = 173{,}500 \text{ miles/sec}$$

38. α and β as Functions of L, C, R, and G. It has been seen that $\gamma = \alpha + j\beta = \sqrt{zy}$, that $z = R + j\omega L$, and $y = G + j\omega C$. In these equations it must be remembered that L, C, R, and G are for the unit length under consideration, here taken as 1 mile. Writing out $\sqrt{zy}$

$$\alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad [5\text{–}29]$$

Let both sides of this equation be squared:

$$\begin{aligned} \alpha^2 - \beta^2 + j2\alpha\beta &= (R + j\omega L)(G + j\omega C) \\ &= (RG - \omega^2 LC) + j(G\omega L + R\omega C) \end{aligned}$$

By equating real and quadrature components there are obtained two equations from which α and β can be found:

$$\alpha^2 - \beta^2 = RG - \omega^2 LC \qquad [5\text{–}30]$$

$$2\alpha\beta = G\omega L + R\omega C \qquad [5\text{–}31]$$

Solving equation 5–31 for α and substituting into equation 5–30

$$\alpha = \frac{G\omega L + R\omega C}{2\beta}$$

and

$$\frac{(G\omega L + R\omega C)^2}{4\beta^2} - \beta^2 = RG - \omega^2 LC$$

or

$$(G\omega L + R\omega C)^2 - 4\beta^4 = (RG - \omega^2 LC)4\beta^2$$

or

$$\beta^4 + \beta^2(RG - \omega^2 LC) - \frac{(G\omega L + R\omega C)^2}{4} = 0$$

This is a quadratic equation in β^2 and can be solved readily giving

$$\beta = \sqrt{\tfrac{1}{2}[\sqrt{(R^2 + \omega^2L^2)(G^2 + \omega^2C^2)} - (RG - \omega^2LC)]} \quad [5\text{–}32]$$

In a similar manner α can be found:

$$\alpha = \sqrt{\tfrac{1}{2}[\sqrt{(R^2 + \omega^2L^2)(G^2 + \omega^2C^2)} + (RG - \omega^2LC)]} \quad [5\text{–}33]$$

The equations 5–32 and 5–33 will be useful later in the treatment of special lines.

39. Illustrative Example. The values of α and β will be determined for the line of Art. 33 employing the parameters L, C, R, and G. For the calculation of α and β the following are needed:

$$R^2 = 103 \qquad (\omega L)^2 = 386$$

$$G^2 = 0.084 \times 10^{-12} \qquad (\omega C)^2 = 1588 \times 10^{-12}$$

$$RG = 2.94 \times 10^{-6} \qquad \omega^2LC = 783 \times 10^{-6}$$

Substituting these values into equation 5–33 gives

$$\alpha = \sqrt{\tfrac{1}{2}[\sqrt{489 \times (1588 \times 10^{-12})} - 780 \times 10^{-6}]}$$

$$= \sqrt{\tfrac{1}{2}(881 \times 10^{-6} - 780 \times 10^{-6})} = 0.00712 \text{ neper/mile}$$

An inspection of equations 5–32 and 5–33 shows that β may be obtained by changing the sign of the last term in the above equation for α; hence

$$\beta = \sqrt{\tfrac{1}{2}(881 \times 10^{-6} + 780 \times 10^{-6})} = 0.0288 \text{ radian/mile}$$

It is seen that these are the same values as obtained for α and β by the method given in Art. 35.

40. Special Lines. (*a*) *Cables.* In most cable circuits for use at voice frequencies the inductive reactance is low compared to the resistance, and the leakage conductance is negligible with respect to the capacity susceptance. The inductance is very low owing to the proximity of the conductors, which causes the external flux linkages to be very small. (See equation 1–11.) The conductance is very small if the cable is well constructed with good insulation. Under these conditions, approximations can be made which will greatly simplify the calculations. Thus

$$z = R + j\omega L \doteq R$$

$$y = G + j\omega C \doteq j\omega C$$

For these conditions equations 5–32 and 5–33 become

$$\beta = \sqrt{\tfrac{1}{2}\sqrt{R^2\omega^2C^2}}$$

$$= \sqrt{\frac{R\omega C}{2}} \qquad [5\text{–}34]$$

$$\alpha = \sqrt{\frac{R\omega C}{2}} \qquad [5\text{–}35]$$

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{R}{j\omega C}} = \sqrt{\frac{-jR}{\omega C}}$$

$$= \sqrt{-j}\sqrt{\frac{R}{\omega C}} = \sqrt{\frac{R}{\omega C}}\underline{/-45^\circ} \qquad [5\text{–}36]$$

$$\lambda = 2\pi\sqrt{\frac{2}{R\omega C}}$$

and

$$v = \frac{\omega}{\beta} = \omega\sqrt{\frac{2}{R\omega C}} = \sqrt{\frac{2\omega}{RC}} \qquad [5\text{–}37]$$

The above equations for v, α, and β show that each is proportional to the square root of the frequency transmitted; thus the higher-frequency portions of the band arrive earlier and are reduced more in magnitude than the low-frequency portions. The phase distortion occurs since β, in equation 5–34, is not directly proportional to ω.

Since the RC cable exhibits both frequency and phase distortion, it is not suitable for the transmission of a wide frequency band.

It is well to note also that, since this is an infinite line, the input impedance is

$$Z_s = Z_0 = \sqrt{\frac{R}{\omega C}}\underline{/-45^\circ}$$

which has a fixed angle of negative 45° regardless of frequency. Thus the current at the input leads the voltage by 45°.

(*b*) *Distortionless Transmission.* An optimum state of affairs would be represented by an α which is independent of frequency and of as low a value as possible, and a β proportional to frequency so that v would be independent of frequency. The determination of the line parameters for such a condition would be rather difficult if attempted by any straightforward development. However, a rearrangement of equations

5–32 and 5–33 leads to the following:

$$\beta = \sqrt{\tfrac{1}{2}[\sqrt{(RG + \omega^2 LC)^2 + (LG - RC)^2\omega^2} - (RG - \omega^2 LC)]} \quad [5\text{–}38]$$

$$\alpha = \sqrt{\tfrac{1}{2}[\sqrt{(RG + \omega^2 LC)^2 + (LG - RC)^2\omega^2} + (RG - \omega^2 LC)]} \quad [5\text{–}39]$$

In these equations it is seen that a considerable simplification will result if the condition that $LG = RC$ exists. Then

$$\beta = \sqrt{\tfrac{1}{2}[(RG + \omega^2 LC) - (RG - \omega^2 LC)]}$$

$$= \sqrt{\omega^2 LC} = \omega\sqrt{LC} \quad [5\text{–}40]$$

and

$$\alpha = \sqrt{\tfrac{1}{2}[(RG + \omega^2 LC) + (RG - \omega^2 LC)]}$$

$$= \sqrt{RG} \quad [5\text{–}41]$$

Thus α and β reduce not only to very simple forms but also to precisely the functions which were desired: α is independent of frequency, and β is proportional to frequency. From this

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \quad [5\text{–}42]$$

The assumption $LG = RC$ is the same as $L/R = C/G$. In other words $\boldsymbol{z}(= R + j\omega L)$ and $\boldsymbol{y}(= G + j\omega C)$ have the same angle.

$\boldsymbol{Z}_0$ can be calculated from

$$\boldsymbol{Z}_0 = \sqrt{\frac{\boldsymbol{z}}{\boldsymbol{y}}}$$

$$\boldsymbol{Z}_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{R + j\omega L}{G + j\dfrac{\omega LG}{R}}}$$

$$= \sqrt{\frac{R(R + j\omega L)}{G(R + j\omega L)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} \quad [5\text{–}43]$$

The condition of $L/R = C/G$ for the above relatively perfect case is difficult to attain because G is generally very small on well-built lines, making it necessary for L to be very large or for R to be small. R cannot be decreased sufficiently because of economic limitations, and it is not advisable to increase G artificially owing to its effect on α. Thus on an actual line it may be suspected from the above discussion that artificially increasing the inductance would have the effect of decreasing R and thus

should improve the attenuation constant. This addition of inductance is called *loading*.

41. Elements of High-Frequency Coaxial-Cable Transmission. A type of line which is of very great importance at high frequencies is the coaxial cable. The equations for L and C of such a line have already been written. They are

$$L = 0.741 \log \frac{b}{a} \times 10^{-3} \text{ henry/mile} \qquad [1\text{–}25]$$

$$C = \frac{0.0388}{\log \frac{b}{a}} \ \mu\text{f/mile} \qquad [1\text{–}37]$$

and

$$R_{ac} = 67.5\sqrt{f}\left(\frac{1}{a} + \frac{1}{b}\right) \times 10^{-4} \text{ ohm/mile} \qquad [1\text{–}84]$$

where ϵ_r is now taken as unity, a and b are in centimeters, and f is expressed in cycles per second.

At very high frequencies G can be neglected, and ωL will be very large in comparison with R. Hence at high frequencies

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \doteq \sqrt{\frac{L}{C}} \qquad [5\text{–}44]$$

With these assumptions, α of equation 5–33 can be written

$$\alpha = \sqrt{\tfrac{1}{2}[\sqrt{\omega^2C^2(R^2 + \omega^2L^2)} - \omega^2LC]}$$

$$= \sqrt{\frac{\omega C}{2}\left[\omega L\sqrt{1 + \frac{R^2}{\omega^2L^2}} - \omega L\right]}$$

$$= \sqrt{\frac{\omega C}{2}\left[\omega L\left(1 + \frac{R^2}{2\omega^2L^2}\right) - \omega L\right]} \qquad [5\text{–}45]$$

where $\left(1 + \dfrac{R^2}{\omega^2L^2}\right)^{1/2}$ is expanded by the binomial theorem and all terms above the second degree are dropped. Thus

$$\alpha = \sqrt{\frac{\omega C}{2} \cdot \frac{R^2}{2\omega L}} = \frac{1}{2}\sqrt{\frac{R^2C}{L}}$$

$$= \frac{R}{2}\sqrt{\frac{C}{L}} \doteq \frac{R}{2Z_0} \qquad [5\text{–}46]$$

The attenuation of the line can be written now in terms of dimensions if equations 1–25 and 1–37 are substituted into equation 5–44 and this with equation 1–84 into 5–46.

$$Z_0 \doteq \sqrt{\frac{0.741 \log \frac{b}{a} \times 10^{-3}}{\frac{0.0388}{\log \frac{b}{a}} \times 10^{-6}}}$$

$$\doteq 138 \log \frac{b}{a} \text{ ohms} \qquad [5\text{–}47]$$

α then becomes

$$\alpha = 24.4 \times 10^{-6} \frac{\sqrt{f}\left(\frac{1}{a} + \frac{1}{b}\right)}{\log \frac{b}{a}} \text{ nepers/mile} \qquad [5\text{–}48]$$

Thus α is given in terms of a and b. The derivative of this function with respect to a may be set equal to zero to obtain a value of a in terms of b which will produce a minimum value of α. In this derivative b must be considered as a constant. It is found that for a minimum α, $b/a = 3.6$, in which case Z_0 becomes about 77 ohms.

It is seen that once a and b are fixed the attenuation will be proportional to the square root of the frequency. Also it will be noted that for coaxial lines the characteristic impedance is relatively low, being about 75 ohms compared with the value of 745 ohms obtained for the aerial line discussed in Art. 33. The velocity of transmission on such a line is very near to the velocity of light, as can be easily seen from the following equations. It is only necessary to change the sign of the last term in equation 5–45 in order to obtain β. Thus

$$\beta = \sqrt{\frac{\omega^2 LC}{2}\left(2 + \frac{R^2}{2\omega^2 L^2}\right)}$$

$$= \omega\sqrt{LC}\sqrt{1 + \frac{R^2}{4\omega^2 L^2}}$$

$$\doteq \omega\sqrt{LC}\left(1 + \frac{R^2}{8\omega^2 L^2}\right)$$

From this

$$v = \frac{\omega}{\beta} \doteq \frac{1}{\sqrt{LC}} \cdot \frac{1}{1 + \dfrac{R^2}{8\omega^2 L^2}}$$

This shows that the velocity will be slightly less than $v' = 1/\sqrt{LC}$. Substitute equations 1–25 and 1–37 into this equation for v'. Then

$$v' = \frac{1}{\left[0.741 \log \dfrac{b}{a} \times 10^{-3} \times \dfrac{0.0388 \times 10^{-6}}{\log \dfrac{b}{a}}\right]^{1/2}}$$

$$\doteq 186{,}285 \text{ miles/sec}$$

or using equations 1–24 and 1–35, $v' \doteq 3 \times 10^8$ meters per sec. If R were actually zero and there were no leakage conductance, as assumed, then the maximum velocity this transmission could have would be the velocity of light.

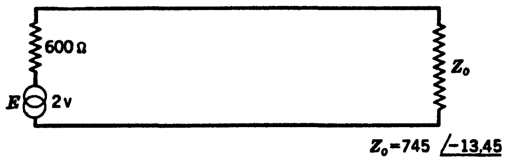


FIG. 5–3. For use in connection with illustrative example of Art. 42.

42. The General Line and the Effect of Loading. The values of Z_0, α, and β as obtained from the foregoing calculations will be applied to the 200-mile line. Assume that the line is terminated at the receiving end in Z_0, and a generator (796 cycles per second) with an emf of 2 volts and internal resistance of 600 ohms is connected to the sending end. (See Fig. 5–3.)

$$Z_0 = 745\underline{/-13.45°} = 725 - j173 \text{ ohms}$$

$$I_s = \frac{E_g}{R_g + Z_0} = \frac{2}{600 + 725 - j173} = \frac{2}{1335\underline{/-7.45°}}$$

$$= 0.0015\underline{/7.45°} \text{ ampere}$$

The total nepers attenuation of the line is

$$N = 200 \times 0.00712 = 1.424$$

$$\frac{I_s}{I_r} = \epsilon^{1.424} = 4.16$$

Therefore the received current, since the line is correctly terminated, is

$$I_r = \frac{I_s}{4.16} = \frac{0.0015}{4.16} = 0.000360 \text{ ampere}$$

The decibel attenuation of the line is easily found from the current ratio,

$$\text{db} = 20 \log 4.16 = 20 \times 0.619 = 12.38$$

The phase rotation of the current will be $\theta = 200 \times 0.0288 = 5.76$ radians or $\theta = 330°$.

Let it be recalled again that the effect of γ is to multiply the sending-end current, or voltage, by a factor

$$\epsilon^{-\gamma l} = \epsilon^{-\alpha l}\epsilon^{-j\beta l} = \epsilon^{-N}\epsilon^{-j\theta} = \epsilon^{-N}\underline{/-\theta°}$$

As was pointed out previously, the factor ϵ^{-N} merely decreases the magnitude of the sending-end quantity, and the factor $\epsilon^{-j\theta}$ ($= \cos\theta - j\sin\theta$) rotates the vector of the quantity through an angle θ in the clockwise or *negative* direction. Thus the receiving-end current has an angle,

$$7.45° - 330° = -322.55°$$

Therefore the receiving-end current is $0.000360\underline{/-322.55°}$ ampere.

The received power is the power lost in the resistance component of the terminating impedance. It is seen above that the resistance component of Z_0 is 725 ohms. Thus

$$\begin{aligned} P_r = I_r^2R_r &= (3.60 \times 10^{-4})^2 \cdot 725 \\ &= 94.0 \times 10^{-6} \text{ watt} \\ &= 94.0\ \mu\text{w} \end{aligned}$$

In order to consider a loaded line it will be assumed that an inductance coil with $R = 7.3$ ohms and $L = 246$ mh is placed in the line at intervals of 7.88 miles. At low frequencies this inductance can be considered as uniformly distributed so that the added inductance per mile is $246/7.88 = 31.25$ mh and the added resistance per mile is $7.3/7.88 = 0.927$ ohm. Thus the fundamental parameters become

$$\begin{aligned} R &= 10.15 + 0.927 = 11.08 \text{ ohms/mile} \\ L &= 3.93 + 31.25 = 35.18 \text{ mh/mile} \\ C &= 0.00797\ \mu\text{f/mile} \\ G &= 0.29 \times 10^{-6} \text{ mho/mile} \end{aligned}$$

Again calculating Z_0

$$\begin{aligned} z &= 11.08 + j5000 \times 0.03518 \\ &= 11.08 + j176 = 176\underline{/86.39°} \text{ ohms} \end{aligned}$$

From Art. 33,

$$y = 39.85 \times 10^{-6} \underline{/89.58^\circ} \text{ mho}$$

Hence

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{176}{39.85 \times 10^{-6}} \underline{/-3.19^\circ}}$$

$$= 2102 \underline{/-1.59^\circ} = 2102 - j58.4 \text{ ohms}$$

For the calculation of α and β:

$$R^2 = 122.8 \qquad (\omega L)^2 = 31{,}000$$

$$G^2 = 0.084 \times 10^{-12} \qquad (\omega C)^2 = 1588 \times 10^{-12}$$

$$RG = 3.21 \times 10^{-6} \qquad \omega^2 LC = 7013 \times 10^{-6}$$

Substituting into equations 5–32 and 5–33

$$\beta = \sqrt{\tfrac{1}{2}[\sqrt{31{,}123 \times 1588 \times 10^{-12}} + 7010 \times 10^{-6}]}$$

$$= \sqrt{\tfrac{1}{2}(7030 + 7010) \times 10^{-6}}$$

$$= 0.0838 \text{ radian/mile}$$

$$\alpha = \sqrt{\tfrac{1}{2}(7030 - 7010) \times 10^{-6}}$$

$$= 0.0032 \text{ neper/mile}^2$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.0838} = 75.0 \text{ miles}$$

$$v = \frac{\omega}{\beta} = \frac{5000}{0.0838} = 59{,}700 \text{ miles/sec}$$

Applying these results to the 200-mile line in the same manner as for the nonloaded line, the input current becomes

$$I_s = \frac{2}{600 + 2102 - j58.4}$$

$$= \frac{2}{2702 \underline{/-1.235^\circ}} = 0.000740 \underline{/1.235^\circ} \text{ ampere}$$

Now calculate $\epsilon^{-\gamma l}$, and operate directly with it on I_s.

$$\epsilon^{-\gamma l} = \epsilon^{-\alpha l} \epsilon^{-j\beta l} = \epsilon^{-\alpha l} \underline{/-\beta l}$$

[2] When the value of α is very low it is better to calculate it by determining the real part of $\sqrt{zy}$ directly.

$$\epsilon^{-\alpha l}\underline{/-\beta l} = \epsilon^{-0.0032\times 200}\underline{/-0.0838 \times 200}$$
$$= \epsilon^{-0.64}\underline{/-16.76 \text{ rad.}}$$
$$= \frac{1}{1.898}\underline{/-960^\circ} = 0.527\underline{/-960^\circ}$$

Thus $I_r = 0.000740\underline{/1.235^\circ} \times 0.527\underline{/-960^\circ} = 0.000390\underline{/-958.76^\circ}$ ampere. The resistance component of the terminating Z_0 is 2102 ohms. Thus the power delivered is

$$P_r = (0.000390)^2 \times 2102$$
$$= 320 \text{ microwatts}$$

The decibel attenuation of the line is

$$\text{db} = 20 \log 1.898 = 20 \times 0.2780 = 5.56$$

A comparative summary of the two lines follows:

	NONLOADED	LOADED
Z_0	$745\underline{/-13.45^\circ}$ ohms	$2102\underline{/-1.59^\circ}$ ohms
α	0.00712 neper/mile	0.0032 neper/mile
β	0.0288 radian/mile	0.0838 radian/mile
λ	218 miles	75.0 miles
v	173,500 miles/sec	59,700 miles/sec
I_s	$0.0015\underline{/7.45^\circ}$ ampere	$0.000740\underline{/1.235^\circ}$ ampere
I_r	$0.000360\underline{/-322.55^\circ}$ ampere	$0.000390\underline{/-958.76^\circ}$ ampere
db	12.38	5.56
P_r	94.0 microwatts	320 microwatts
βl	5.76 radians	16.76 radians

The important item to note in the above tabulation is that the attenuation for the loaded line is approximately one-half that for the nonloaded line. The power delivered over the loaded line is about 3.5 times the power over the nonloaded line. However, loading has caused a great decrease in the phase velocity.

Instead of placing loading coils at intervals along the line, the effect may be obtained by wrapping the conductors with iron or permalloy tape which uniformly distributes the added inductance.

PROBLEMS

Make all calculations for Probs. 5-1 to 5-6 inclusive at 796 cycles per second.

5-1. Given a line A which has the following parameters per loop mile:

$$R = 4.02 \text{ ohms}$$
$$L = 3.37 \text{ mh}$$
$$C = 0.00898\ \mu\text{f}$$
$$G = 5 \times 10^{-6} \text{ mho}$$

This line is 200 miles long and is terminated in its Z_0. Calculate Z_0, α, β, v, and λ.

/ **5–2.** A generator of 500 ohms internal *resistance* and 10 volts generated emf is connected to the sending end of line A. Find the voltage across the line and the current in the line at the receiving and sending ends and at intervals of 25 miles along the line. Plot results using both polar and cartesian coordinates.

5–3. Given a line B with the following parameters per loop mile:

$$R = 5.08 \text{ ohms}$$

$$L = 2.22 \text{ mh}$$

$$C = 0.01454 \ \mu\text{f}$$

$$G = 0.58 \times 10^{-6} \text{ mho}$$

(*a*) Calculate $\boldsymbol{Z}_0$, α, β, v, and λ.

(*b*) Load the line with an additional $L = 246$ mh and $R = 7.3$ ohms at intervals of 7.88 miles, and recalculate $\boldsymbol{Z}_0$, α, β, v, and λ. Note any differences caused by loading.

5–4. Calculate the decibel attenuation of 200 miles of line B when properly terminated, when both loaded and unloaded.

5–5. A cable line C has the following parameters per loop mile:

$$R = 42.1 \text{ ohms}$$

$$L = 1 \text{ mh}$$

$$C = 0.062 \ \mu\text{f}$$

$$G = 1.5 \times 10^{-6} \text{ mho}$$

Calculate $\boldsymbol{Z}_0$, α, β, v, and λ.

5–6. Loading is added to the line C at intervals of 1.135 miles. The additional R and L for this interval are 2.7 ohms and 31 millihenrys respectively. Recalculate $\boldsymbol{Z}_0$, α, β, v, and λ.

5–7. (*a*) Calculate and plot curves showing the relationship between attenuation constant versus frequency and between wavelength constant versus frequency for the cable line of Prob. 5–5 over a frequency range of 100 to 5000 cycles per second.

(*b*) Calculate and plot curves showing the relationship between characteristic impedance versus frequency, velocity of propagation versus frequency, and wavelength versus frequency for the above line and frequency range.

5–8. A parallel-wire line is composed of two No. 10 AWG hard-drawn copper conductors separated center to center by a distance of 5 inches. Determine the characteristic impedance of the line at 1000 kilocycles:

(*a*) When resistance and shunt conductance are neglected,

(*b*) When shunt conductance only is neglected.

5–9. The line of Prob. 5–8 is 100 meters long. Determine α and β, the decibel loss for the line, and the total change in phase of the current in degrees:

(*a*) When shunt conductance is neglected,

(*b*) When shunt conductance is taken as 3.82×10^{-6} mho per meter.

CHAPTER VI

OPEN-CIRCUIT AND SHORT-CIRCUIT LINES

The previous chapter treated the uniform line which was properly terminated. Such a line led to the condition under which the propagation was unidirectional, that is, no reflected wave ever could return from the far end. Everyone has had experience with water waves in channels or elastic-displacement waves on long stretched wires, and it is well known that, if there are irregularities in either, a reflection results. If a wave is sent from one end of a long wire fastened firmly at the far end, in a short time a wave returns. This returned wave is the result of the discontinuity at the distant end. Echoes are another illustration of reflection caused by irregularity or discontinuity of the conducting medium. On account of the similarity between the electric wave on a transmission line and the waves mentioned above, reflection would be expected in every line which is not infinite (or terminated in Z_0).

The present chapter introduces the problem of reflection and considers the two limiting conditions of mismatch at the receiving end — that of an open circuit and that of a short circuit. The basis is also laid for further development of reflection phenomena and various miscellaneous treatments which are taken up in Chapter VII.

43. General Equations. In the preceding chapter two equations,

$$V = V_s \cosh \sqrt{zy}\, x - I_s Z_0 \sinh \sqrt{zy}\, x \qquad [5\text{–}11]$$

and

$$I = I_s \cosh \sqrt{zy}\, x - \frac{V_s}{Z_0} \sinh \sqrt{zy}\, x \qquad [5\text{–}12]$$

were derived, and on the basis of an infinite line the term $\sqrt{z/y}$ was identified with Z_0. It should, however, be carefully noted that the derivation of equations 5–11 and 5–12 did not depend on the line termination. The only boundary conditions used were at the receiving end where $x = 0$, $V = V_r$, and $I = I_r$. Thus these equations can be easily extended to cover lines terminated in a more general Z_r which may or may not be equal to Z_0. Obviously, of course, in such a line the input impedance is no longer equal to Z_0, but Z_0, being a constant of the line, can be kept in the equation.

Letting the length of the line be represented by S, the above equations will give the receiving-end quantities in terms of S as follows:

$$V_r = V_s \cosh \sqrt{zy}\, S - I_s Z_0 \sinh \sqrt{zy}\, S \qquad [6\text{-}1]$$

$$I_r = I_s \cosh \sqrt{zy}\, S - \frac{V_s}{Z_0} \sinh \sqrt{zy}\, S \qquad [6\text{-}2]$$

Similarly from equations 5–9 and 5–10

$$V_s = V_r \cosh \sqrt{zy}\, S + I_r Z_0 \sinh \sqrt{zy}\, S \qquad [6\text{-}3]$$

$$I_s = I_r \cosh \sqrt{zy}\, S + \frac{V_r}{Z_0} \sinh \sqrt{zy}\, S \qquad [6\text{-}4]$$

These four equations are the general equations of a transmission line and do not involve the terminating impedance explicitly. Therefore the indicated voltages and currents can be considered as existent at any two points on a line separated by a distance S, which in this case is to be taken as the length of the line under consideration.

Keeping in mind that Z_r must equal V_r/I_r, equation 6–3 can be rearranged as follows:

$$I_r = \frac{V_s}{Z_r \cosh \sqrt{zy}\, S + Z_0 \sinh \sqrt{zy}\, S} \qquad [6\text{-}5]$$

and from equation 6–4

$$I_s = I_r \left(\cosh \sqrt{zy}\, S + \frac{Z_r}{Z_0} \sinh \sqrt{zy}\, S\right) \qquad [6\text{-}5a]$$

$$= \frac{V_s (Z_0 \cosh \sqrt{zy}\, S + Z_r \sinh \sqrt{zy}\, S)}{Z_0 (Z_r \cosh \sqrt{zy}\, S + Z_0 \sinh \sqrt{zy}\, S)} \qquad [6\text{-}6]$$

These equations are derived in a different manner in Appendix V.

44. Direct and Reflected Waves. In order to appreciate more fully the purpose of terminating a line in its characteristic impedance, let equations 5–9 and 5–10 be rewritten employing exponentials and the attenuation and phase shift constants. Then

$$V = \frac{V_r + I_r Z_0}{2} \epsilon^{\alpha s_r} \epsilon^{j\beta s_r} + \frac{V_r - I_r Z_0}{2} \epsilon^{-\alpha s_r} \epsilon^{-j\beta s_r} \qquad [6\text{-}7]$$

$$I = \frac{I_r + \dfrac{V_r}{Z_0}}{2} \epsilon^{\alpha s_r} \epsilon^{j\beta s_r} + \frac{I_r - \dfrac{V_r}{Z_0}}{2} \epsilon^{-\alpha s_r} \epsilon^{-j\beta s_r} \qquad [6\text{-}8]$$

where s_r will be used to indicate distance along a line of length S from the receiving end. Since $V_r = I_r Z_r$

$$V = I_r \frac{Z_r + Z_0}{2} \epsilon^{\alpha s_r} \epsilon^{j\beta s_r} + I_r \frac{Z_r - Z_0}{2} \epsilon^{-\alpha s_r} \epsilon^{-j\beta s_r} \qquad [6\text{–}9]$$

$$I = \frac{I_r}{Z_0} \left[\frac{Z_r + Z_0}{2} \epsilon^{\alpha s_r} \epsilon^{j\beta s_r} - \frac{Z_r - Z_0}{2} \epsilon^{-\alpha s_r} \epsilon^{-j\beta s} \right] \qquad [6\text{–}10]$$

Equation 6–7 is seen to consist of two terms, the first of which increases in magnitude (due to $\epsilon^{\alpha s_r}$) as we go from the receiving to the sending end or decreases in magnitude proceeding from the sending to the receiving end. This term, when associated with the time variation of voltage (which has been canceled out of equation 6–7), results in a voltage wave which originates at the sending end of the line and is often referred to as the direct wave. The operator $\epsilon^{j\beta s_r}$ causes the direct wave to advance in phase from its position at the receiver as we proceed from the receiving end.

The second term of equation 6–7 decreases in magnitude (due to $\epsilon^{-\alpha s_r}$) as we go from the receiving to the sending end. Hence this wave may be thought of as originating at the receiving end, and as such it is called the reflected wave. The operator $\epsilon^{-j\beta s_r}$ causes the reflected wave to drop back in phase from its receiver position.

The voltage at any point on the line is the sum of the two terms shown in equation 6–7. Since the actual time variation of the voltage has been eliminated from this equation, the two terms combine to form a *standing wave*, that is, a plot of V versus s_r.

The loci of the direct and reflected voltage components for a case in which $Z_r < Z_0$ are shown in Fig. 6–1 together with the resultant standing wave. The resultant wave is the vector sum of the direct and reflected components. Since $Z_r < Z_0$, the reflected wave appears as a negative wave with respect to the direct wave. If Z_r were greater than Z_0, then the reflected wave would appear as a positive wave. (See Fig. 6–9 for the open-circuit case of $Z_r = \infty$.) At the load the resultant wave is the arithmetic difference between the direct and reflected waves, whereas at the 90° point it is equal to the arithmetic sum of the two components. At the 180° point the resultant is again the difference and at 270° again the sum. In rectangular coordinates, the loci plot as shown in Fig. 6–2.

Examination of equation 6–9 shows that when $Z_r = Z_0$ (the line properly terminated) the reflected component disappears. The loci of the voltage wave for the matched case are shown in Fig. 6–3 in polar coordinates and in Fig. 6–4 in rectangular coordinates. It is to be noted

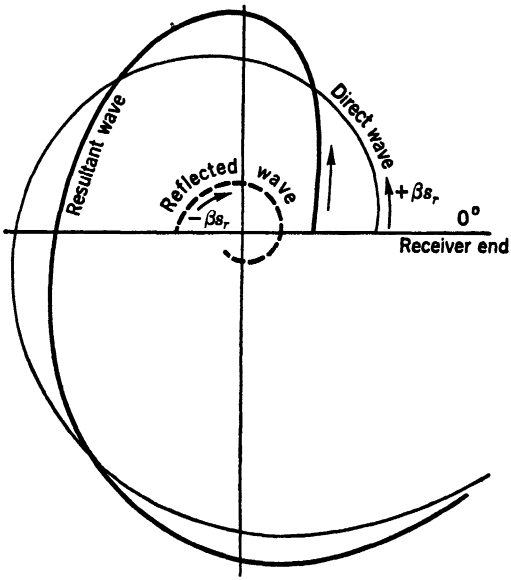


FIG. 6-1. Voltage loci for mismatched condition. $Z_r < Z_0$. $K = 0.5\underline{/180°}$ (See page 211.)

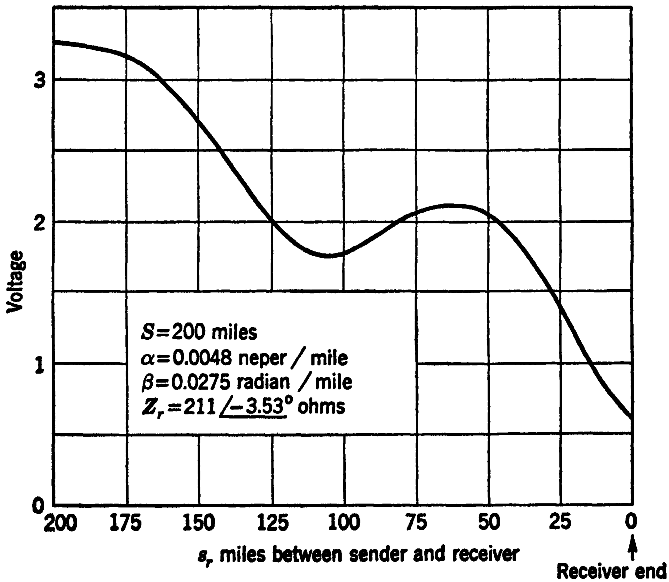


FIG. 6-2. Voltage along mismatched line showing the standing wave. (See Fig. 6-1.)

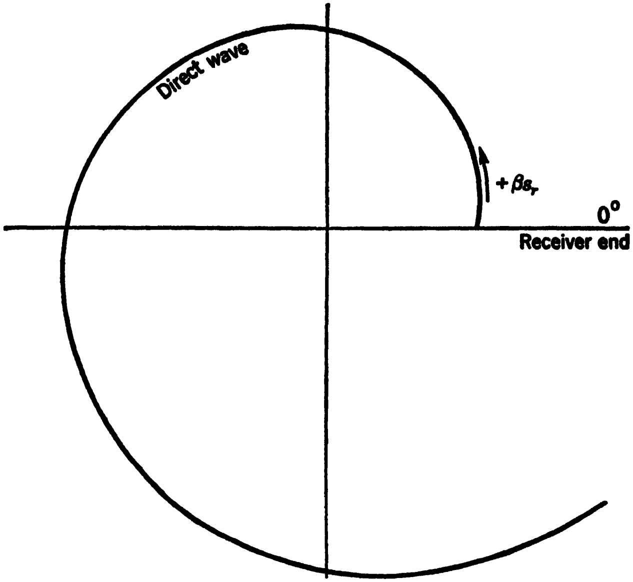


FIG. 6-3. Voltage loci for matched condition. No reflected wave.

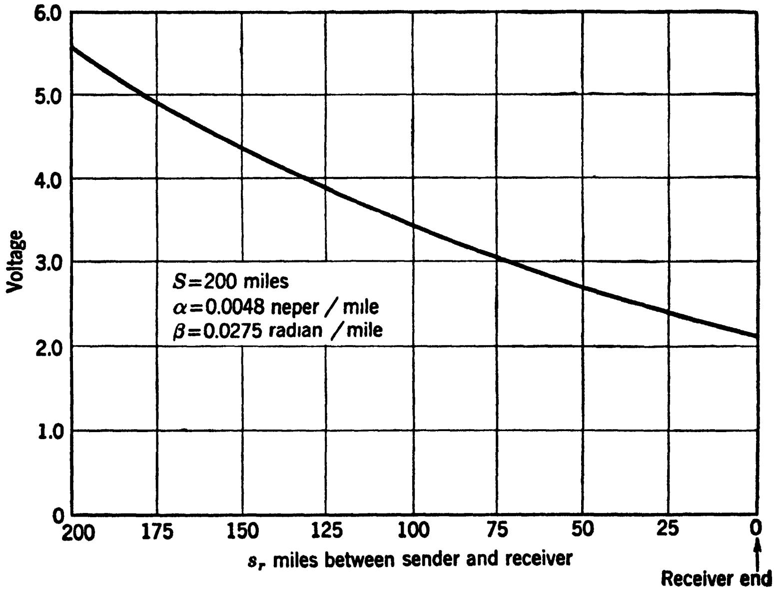


FIG. 6-4. Voltage along matched line. (See Fig. 6-3.)

that the spiral is smooth, the voltage variation is uniform, and no standing waves are present under these conditions.

45. Sending and Transfer Impedances. The *sending-end*, or input, *impedance* of the line has not yet been mentioned. However, in any practical treatment of the line its value is necessary in order to determine the input current when a given generator voltage is applied to the sending end. Equation 6–6 gives this impedance directly by transposition. Thus

$$Z_s = \frac{V_s}{I_s} = Z_0 \left[\frac{Z_r \cosh \sqrt{zy}\, S + Z_0 \sinh \sqrt{zy}\, S}{Z_0 \cosh \sqrt{zy}\, S + Z_r \sinh \sqrt{zy}\, S}\right] \qquad [6\text{–}11]$$

It is seen immediately that, if $Z_r = Z_0$, the input impedance reduces to Z_0 as it should.

The *sending-end transfer impedance* of a line is defined as the ratio V_s/I_r, and using equation 6–5 this becomes

$$Z_{tr} = Z_r \cosh \sqrt{zy}\, S + Z_0 \sinh \sqrt{zy}\, S \qquad [6\text{–}12]$$

46. Input Impedance, Open- and Short-Circuit Line. If Z_r becomes infinite, an open-circuit line results. The reflection which takes place under this condition is characterized by the fact that the current I_r at the end of the line is zero. In such a line the input impedance varies widely as either the length of the line or the frequency of the transmitted wave is changed.

If numerator and denominator of equation 6–11 are divided by Z_r, and Z_r allowed to approach an infinitely large value (open circuit), the input impedance of the open-circuit line is

$$Z_{so} = Z_0 \frac{\cosh \sqrt{zy}\, S}{\sinh \sqrt{zy}\, S} \qquad [6\text{–}13]$$

Thus

$$Z_{so} = Z_0 \coth \sqrt{zy}\, S = Z_0 \frac{\sqrt{\sinh^2 a + \cos^2 b} \,\Big/ \tan^{-1} \dfrac{\sin b \sinh a}{\cos b \cosh a}}{\sqrt{\sinh^2 a + \sin^2 b} \,\Big/ \tan^{-1} \dfrac{\sin b \cosh a}{\cos b \sinh a}} \qquad [6\text{–}14]$$

where $a = \alpha S$ and $b = \beta S$ from $\sqrt{zy}\, S = a + jb$.

The input current to the line becomes

$$I_{so} = \frac{V_s}{Z_{so}} = \frac{V_s \tanh \sqrt{zy}\, S}{Z_0} \qquad [6\text{–}15]$$

For the short-circuit line it is only necessary to set $Z_r = 0$, or what amounts to the same thing, $V_r = 0$. The input impedance can again be determined from equation 6–11 as

$$Z_{ss} = Z_0 \frac{\sinh \sqrt{zy}\, S}{\cosh \sqrt{zy}\, S} = Z_0 \tanh \sqrt{zy}\, S \qquad [6\text{–}16]$$

or

$$Z_{ss} = Z_0 \frac{\sqrt{\sinh^2 a + \sin^2 b} \Big/ \tan^{-1} \dfrac{\sin b \cosh a}{\cos b \sinh a}}{\sqrt{\sinh^2 a + \cos^2 b} \Big/ \tan^{-1} \dfrac{\sin b \sinh a}{\cos b \cosh a}} \qquad [6\text{–}17]$$

Also

$$I_{ss} = \frac{V_s}{Z_{ss}} = \frac{V_s}{Z_0 \tanh \sqrt{zy}\, S} \qquad [6\text{–}18]$$

It will be seen from equations 6–14 and 6–17 and Fig. 6–5 that the impedances Z_{so} and Z_{ss} fluctuate in value as the length of the line is

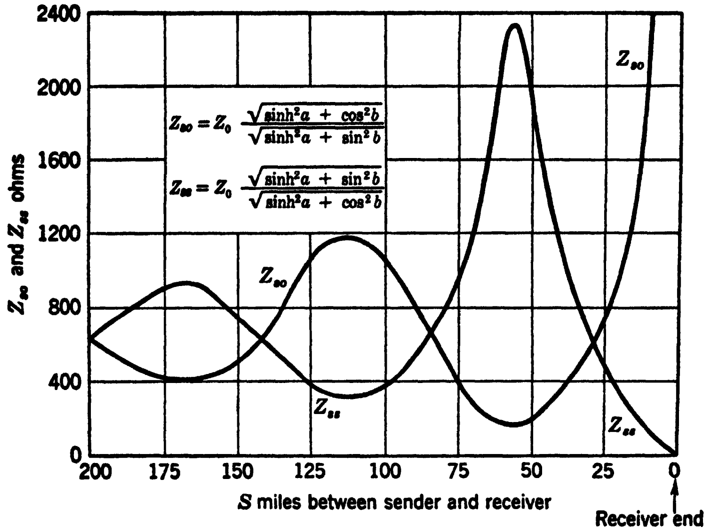


FIG. 6–5. Variation of Z_{so} and Z_{ss} with length of line.

increased. This variation is due to a gradual change in the value of sinh a and a periodic fluctuation in both sin b and cos b. If the sinh a term is not so great as to mask the fluctuation caused by the circular functions, then maximum (or minimum) values of both Z_{ss} and Z_{so} will be passed through at approximately every half wavelength. This follows from the fact that b in these equations is βS and, if S is expressed

in terms of the number of wavelengths n, b becomes $2\pi n$. Z_{so} will be maximum at $n = 0$, whereas Z_{ss} is a minimum. The $\sinh^2 a$ term merely increases gradually and eventually predominates over the fluctuation caused by the circular functions. Continuing with the increase in n, when n becomes $\frac{1}{4}$, b is $\pi/2$, and Z_{so} is a minimum while Z_{ss} is a maximum. Again at $n = \frac{1}{2}$, Z_{so} will be a maximum and Z_{ss} a minimum, because of the circular function being squared. Thus two maxima (or two minima) may be encountered in one-half wavelength.

To illustrate a practical case, curves are plotted in Fig. 6–5 for open- and short-circuit impedances of a nonloaded line versus length of line. Here $Z_0 = 619\underline{/-3.53°}$ ohms, $\alpha = 0.0048$ neper/mile, $\beta = 0.0275$ radian/mile, and $\lambda = 228.5$ miles. It is seen that maxima are passed through every 114.2 miles.

47. Voltage at Open-Circuit End. It is of interest to determine the voltage at the open-circuit end of the line wherein $I_r = 0$. Writing equation 6–1 in terms of open-circuit notation

$$V_{ro} = V_s \cosh \sqrt{zy}\, S - I_{so} Z_0 \sinh \sqrt{zy}\, S$$

Upon substituting for I_{so} the value as given in equation 6–15, there results

$$V_{ro} = V_s \cosh \sqrt{zy}\, S - V_s \frac{\sinh^2 \sqrt{zy}\, S}{\cosh \sqrt{zy}\, S}$$

$$= \frac{V_s\,(\cosh^2 \sqrt{zy}\, S - \sinh^2 \sqrt{zy}\, S)}{\cosh \sqrt{zy}\, S}$$

$$\frac{V_s}{\cosh \sqrt{zy}\, S} \qquad [6\text{–}19]$$

If $\sqrt{zy}\, S$ is written as $a + jb$, where $a = \alpha S$ and $b = \beta S$, the equation becomes, using equation A–25:

$$V_{ro} = \frac{V_s \underline{/ -\tan^{-1} \dfrac{\sin b \sinh a}{\cos b \cosh a}}}{\sqrt{\sinh^2 a + \cos^2 b}} \qquad [6\text{–}20]$$

This relationship is interesting because of the fact that there may be lengths of the line or frequencies of transmitted waves for which V_{ro} may be greater than V_s. This is called *Ferranti effect.*

Confining attention to the absolute value only of equation 6–20, V_{ro} may be greater than V_s only when

$$\sinh^2 a + \cos^2 b < 1$$

As the length of the line increases, $\sinh^2 a$ increases from zero to ever higher values. At the same time $\cos^2 b$ decreases from 1 to 0 and back to 1, etc. Thus there may be certain relatively short lines where the above condition holds. This must be before $\sinh^2 a = 1$ because the lowest value which $\cos^2 b$ can have is zero.

The limiting condition for Ferranti effect exists when V_{ro} is equal to the sending-end voltage, that is, when the denominator of equation 6–20 is unity. Then

$$\begin{aligned} \sinh^2 a + \cos^2 b &= 1 \\ \sinh^2 a &= 1 - \cos^2 b \\ \sinh^2 a &= \sin^2 b \\ \pm \sinh a &= \pm \sin b \\ \pm \sinh \alpha S &= \pm \sin \beta S \end{aligned} \qquad [6\text{–}21]$$

If α and β are such that the line is subject to Ferranti effect, then Ferranti effect will occur for all values of line lengths S for which equation 6–21 is satisfied.

The limiting condition necessary for the occurrence of Ferranti effect is to be found by applying equation 6–21 to very short lines. For low values of S, $\sinh \alpha S \doteq \alpha S$, and $\sin \beta S \doteq \beta S$. Accordingly the limit given by equation 6–21 is $\beta = \alpha$. The same result may also be obtained by differentiating the absolute value of equation 6–20 with respect to S, remembering that $a = \alpha S$ and $b = \beta S$.

$$\frac{dV_{ro}}{dS} = \frac{\beta \sin 2\beta S - \alpha \sinh 2\alpha S}{2\,(\sinh^2 \alpha S + \cos^2 \beta S)^{3/2}}$$

The derivative will be positive if $\beta \sin 2\beta S > \alpha \sinh 2\alpha S$, and thus there will be an increase in V_{ro} with an increase in S. For the very short line the derivative will be positive for

$$2\beta^2 S > 2\alpha^2 S$$

or

$$\beta > \alpha$$

From the above derivative it can also be seen that the value of S at which maximum V_{ro} will exist is given by the condition,

$$\beta \sin 2\beta S = \alpha \sinh 2\alpha S$$

The variations of $\sinh^2 a$, $\cos^2 b$, $(\sinh^2 a + \cos^2 b)$, and V_{ro} with line length are illustrated in Fig. 6–7 for a typical line subject to Ferranti effect. It will be noted that prior to $\sinh^2 a$ reaching the value of unity

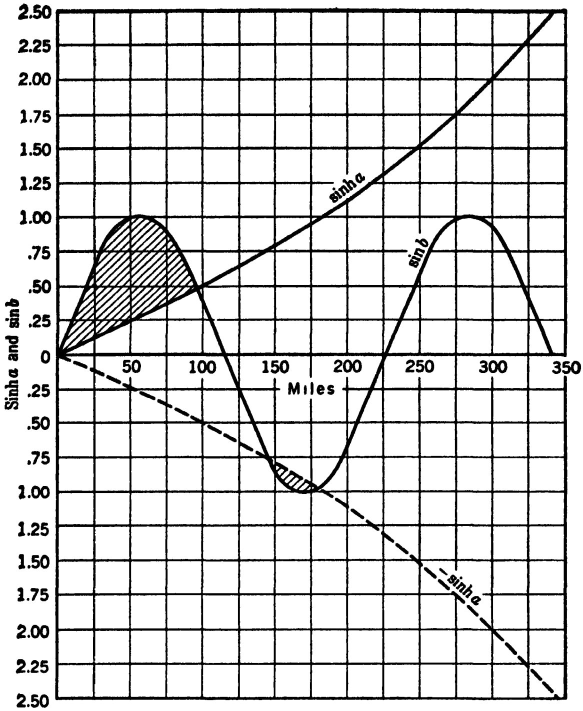


Fig. 6–6. Variation of sinh a and sin b with length of line.

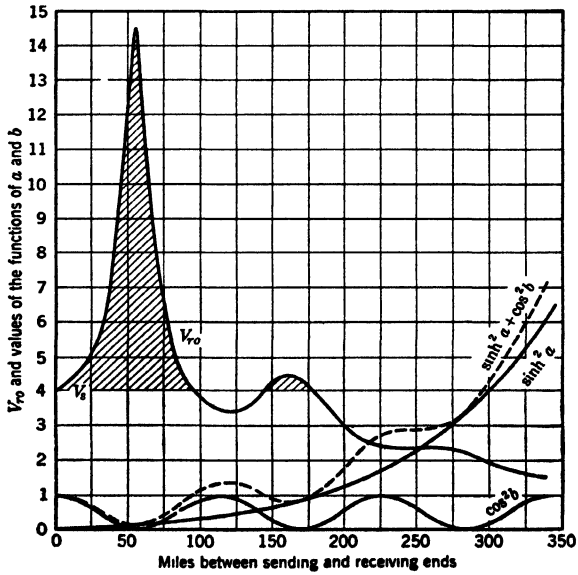


Fig. 6–7. Illustration of Ferranti effect.

the function ($\sinh^2 a + \cos^2 b$) assumes minimum and maximum values. If $\beta \gg \alpha$, as in the illustration, the first minimum value will occur at a point where b is slightly less than $\pi/2$ or at a value of S which is a little less than $\lambda/4$. If α is more nearly equal to β, the $\sinh^2 a$ curve will rise more rapidly with respect to the $\cos^2 b$ curve than that shown in the above illustration, and the minimum value of ($\sinh^2 a + \cos^2 b$) will occur at a lower value of S, thus causing maximum Ferranti effect to occur at a still shorter length of line.

48. Illustrative Example. This discussion of Ferranti effect will be illustrated by calculations on a 165-mil-diameter copper-wire line with 12-inch spacing between wires having the following parameters per loop mile:

$$R = 4.02 \text{ ohms} \qquad C = 0.00898\ \mu\text{f between wires}$$

$$L = 3.37 \text{ mh} \qquad G = 5\ \mu \text{ mhos}$$

At 796 cycles per second

$$\alpha = 0.0048 \text{ neper/mile}$$

$$\beta = 0.0275 \text{ radian/mile}$$

$$\lambda = 228.5 \text{ miles}$$

Equation 6–21 is shown plotted in Fig. 6–6 for values of S from 0 to 341.4 miles or $1\frac{1}{2}\lambda$. It is to be noted from this figure that $\sinh \alpha S$ cuts the $\sin \beta S$ curve at 96.2 miles and that $-\sinh \alpha S$ cuts it at 145.2 miles and again at 179.8 miles. This means that Ferranti effect will occur on lines of the above design of lengths from zero to 96.2 miles and from 145.2 to 179.8 miles. The receiver voltage on open-circuit lines of lengths between 96.2 and 145.2 miles and greater than 179.8 miles will be less than the sending voltage V_s.

In Fig. 6–7 are shown plots of $\sinh^2 a$, $\cos^2 b$, and ($\sinh^2 a + \cos^2 b$) as well as V_{ro} when V_s is 4 volts.

49. Current at Short-Circuit End. The current at the short-circuit end of the line may be found from equation 6–2, using equation 6–18 for $\boldsymbol{I}_s$.

$$I_{rs} = \frac{V_s \cosh \sqrt{zy}\, S}{Z_0 \tanh \sqrt{zy}\, S} - \frac{V_s}{Z_0} \sinh \sqrt{zy}\, S$$

$$= \frac{V_s}{Z_0}\left[\frac{\cosh^2 \sqrt{zy}\, S - \sinh^2 \sqrt{zy}\, S}{\sinh \sqrt{zy}\, S}\right]$$

$$= \frac{V_s}{Z_0 \sinh \sqrt{zy}\, S} \qquad [6\text{–}22]$$

$$= \frac{V_s}{Z_0 \sqrt{\sinh^2 a + \sin^2 b}\ \big/ \tan^{-1} \dfrac{\sin b \cosh a}{\cos b \sinh a}} \qquad [6\text{–}23]$$

When the length of the line is zero, the current is infinite as would be expected for a finite V_s. As the length is increased, $\sinh^2 a$ increases indefinitely, and $\sin^2 b$ fluctuates between zero and unity. Thus the current decreases along a curve which oscillates considerably at first but smooths out as the $\sinh^2 a$ term becomes predominant. The type

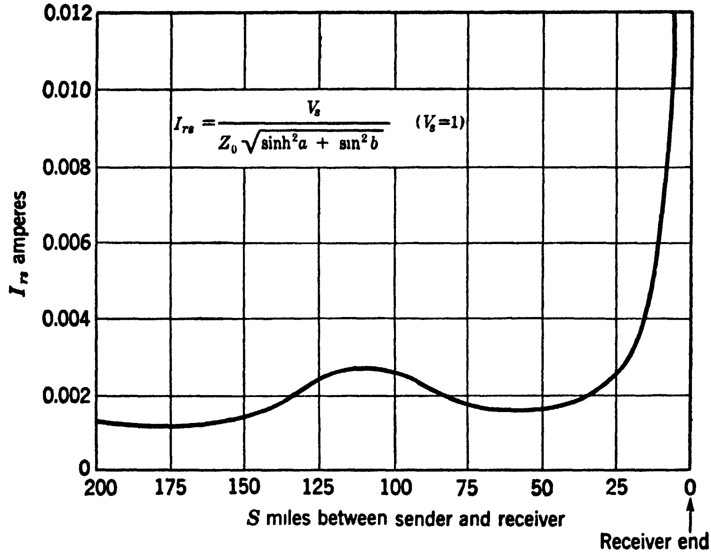


FIG. 6–8. Variation of short-circuit current with length of line.

of oscillation is shown in Fig. 6–8 where equation 6–23 is plotted for lines of various lengths having parameters as given for the nonloaded line of Art. 46 using V_s as $1\underline{/0^\circ}$ volt in each case.

50. Conditions at Any Point on the Open-Circuit Line. The voltage at any point on an open-circuit line in terms of the receiving-end quantities is given by a rearrangement of equation 6–19, such as

$$V = V_{ro} \cosh \sqrt{zy}\, s_r$$

$$= V_{ro} \cosh (\alpha s_r + j\beta s_r)$$

$$= V_{ro}\sqrt{\sinh^2 a_r + \cos^2 b_r}\ \underline{/\tan^{-1} \frac{\sin b_r \sinh a_r}{\cos b_r \cosh a_r}} \qquad [6\text{–}24]$$

where $\sqrt{zy}\, s_r = \alpha s_r + j\beta s_r = a_r + jb_r$.

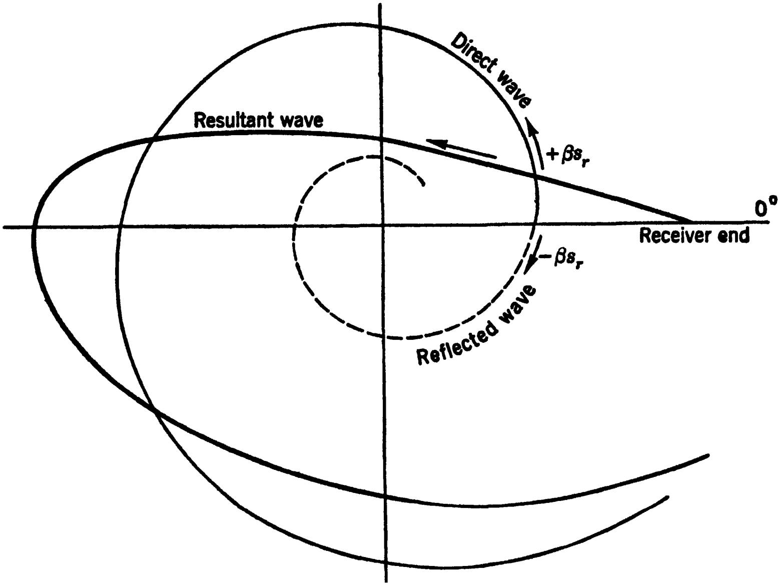


Fig. 6–9. Voltage loci for open-circuit line.

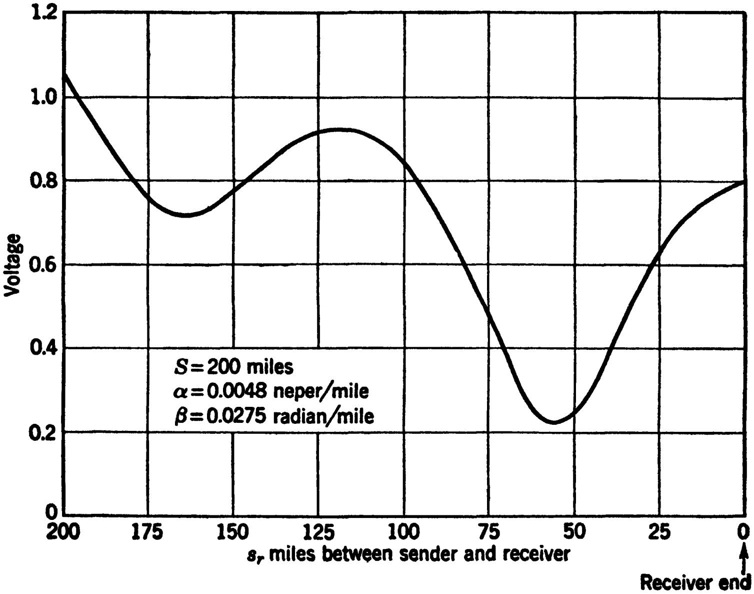


Fig. 6–10. Voltage along open-circuit line. (See Fig. 6–9.)

The current at any point on such a line is given by equation 6–4: ($I_r = 0$).

$$I = \frac{V_{ro}}{Z_0} \sinh \sqrt{zy}\, s_r$$

$$= \frac{V_{ro}}{Z_0} \sqrt{\sinh^2 a_r + \sin^2 b_r} \Big/ \tan^{-1} \frac{\sin b_r \cosh a_r}{\cos b_r \sinh a_r} \qquad [6\text{–}25]$$

The impedance at any point is given by

$$Z_{so} = \frac{V}{I} = Z_0 \coth \sqrt{zy}\, s_r$$

$$= Z_0 \frac{\sqrt{\sinh^2 a_r + \cos^2 b_r} \Big/ \tan^{-1} \dfrac{\sin b_r \sinh a_r}{\cos b_r \cosh a_r}}{\sqrt{\sinh^2 a_r + \sin^2 b_r} \Big/ \tan^{-1} \dfrac{\sin b_r \cosh a_r}{\cos b_r \sinh a_r}} \qquad [6\text{–}26]$$

Note that, in equation 6–26, if s_r is made zero, $Z_{so} = Z_0\sqrt{1/0} = \infty$, which checks with the known condition for an open-circuit line. It will be noticed also that if the length S is substituted into equation 6–26 the equation obtained is the same as equation 6–14.

Writing equations 6–24 and 6–25 in terms of exponentials (or substituting $I_r = 0$ into equations 6–7 and 6–8) equations for V and I at any point on the line are obtained as

$$V = \frac{V_{ro}}{2} \epsilon^{\alpha s_r} \epsilon^{j\beta s_r} + \frac{V_{ro}}{2} \epsilon^{-\alpha s_r} \epsilon^{-j\beta s_r} \qquad [6\text{–}27]$$

$$I = \frac{V_{ro}}{2Z_0} \epsilon^{\alpha s_r} \epsilon^{j\beta s_r} - \frac{V_{ro}}{2Z_0} \epsilon^{-\alpha s_r} \epsilon^{-j\beta s_r} \qquad [6\text{–}28]$$

It is seen that each of these equations is composed of a direct and a reflected wave and that the resultant represents a *standing wave* on the line. In Fig. 6–9 are shown the loci of the direct, the reflected, and the resultant voltage waves for an open-circuit line. It is to be noted that the reflected voltage wave appears as a positive wave. The plot in rectangular coordinates is shown in Fig. 6–10.

If it is desired to express equations 6–24 and 6–25 in terms of sending-end values, it may be done as follows: From equation 6–19

$$V_{ro} = \frac{V_s}{\cosh \sqrt{zy}\, S}$$

and substituting this expression into equation 6–24 gives

$$V = \frac{V_s \cosh \sqrt{zy}\, s_r}{\cosh \sqrt{zy}\, S} \qquad [6\text{–}29]$$

Also from equation 6–25

$$I_s = \frac{V_{ro} \sinh \sqrt{zy}\, S}{Z_0}$$

or

$$V_{ro} = \frac{Z_0 I_s}{\sinh \sqrt{zy}\, S}$$

and

$$I = \frac{Z_0 I_s \sinh \sqrt{zy}\, s_r}{Z_0 \sinh \sqrt{zy}\, S}$$

$$= \frac{I_s \sinh \sqrt{zy}\, s_r}{\sinh \sqrt{zy}\, S} \qquad [6\text{–}30]$$

If α is negligible the equations for V and I may be written as

$$V = \frac{V_s}{\cos \beta S} \cos \beta s_r = V_{ro} \cos \beta s_r \qquad [6\text{–}31]$$

$$I = \frac{I_s}{\sin \beta S} \sin \beta s_r = j \frac{V_{ro}}{Z_0} \sin \beta s_r \qquad [6\text{–}32]$$

From these equations it is seen that V and I are displaced along the line by one-quarter wavelength, and both V and I become zero at points separated by one-half wavelength. Such is the condition which is approached very closely by certain high-frequency lines of comparatively short lengths. It is to be noted from the above that so long as α has a finite value neither the voltage nor current ever reaches the value zero at any point except for the current which is zero at the receiving end.

51. Conditions at Any Point on the Short-Circuit Line. The voltage and current at any point on a short-circuit line may be found from equations 6–3 and 6–4. From equation 6–3

$$V = I_{rs} Z_0 \sinh \sqrt{zy}\, s_r \qquad [6\text{–}33]$$

where s_r is, as before, the distance from the short-circuit end, and I_{rs} is the current through the short circuit. This may be written, by using equation 6–22, as follows:

$$V = \frac{V_s \sinh \sqrt{zy}\, s_r}{\sinh \sqrt{zy}\, S} \qquad [6\text{–}34]$$

Also the current at any point is, by equation 6–4,

$$I = I_{rs} \cosh \sqrt{zy}\, s_r \qquad [6\text{–}35]$$

At the sending end

$$I_s = I_{rs} \cosh \sqrt{zy}\, S$$

or

$$I_{rs} = \frac{I_s}{\cosh \sqrt{zy}\, S}$$

which substituted into equation 6–35 gives I in terms of sending-end values,

$$I = \frac{I_s \cosh \sqrt{zy}\, s_r}{\cosh \sqrt{zy}\, S} \qquad [6\text{–}36]$$

The impedance of the short-circuit line at any point is obtained from equations 6–33 and 6–35 as

$$Z_{ss} = \frac{V}{I} = Z_0 \tanh \sqrt{zy}\, s_r \qquad [6\text{–}37]$$

The equations giving the impedance at any point on the open- and short-circuit lines are the same as equations 6–14 and 6–16 for lines of length $S(= s_r)$. These equations have already been plotted for various line lengths as shown in Fig. 6–5.

By converting equations 6–33 and 6–35 in terms of exponentials (or substituting $V_r = 0$ into equations 6–7 and 6–8), equations for V and I at any point on the line are obtained as

$$V = \frac{I_{rs}Z_0}{2} \epsilon^{\alpha s_r} \epsilon^{j\beta s_r} - \frac{I_{rs}Z_0}{2} \epsilon^{-\alpha s_r} \epsilon^{-j\beta s_r} \qquad [6\text{–}38]$$

$$I = \frac{I_{rs}}{2} \epsilon^{\alpha s_r} \epsilon^{j\beta s_r} + \frac{I_{rs}}{2} \epsilon^{-\alpha s_r} \epsilon^{-j\beta s_r} \qquad [6\text{–}39]$$

Again it is seen that these are standing waves each composed of a direct and a reflected component and that the reflected voltage wave appears as a negative wave. The resultant wave shown in the plot of Fig. 6–11 clearly shows this latter condition. The plot of the standing wave as it appears in rectangular coordinates is shown in Fig. 6–12.

If α is negligible, the equations for V and I become

$$V = \frac{V_s}{\sin \beta S} \sin \beta s_r = jI_{rs}Z_0 \sin \beta s_r \qquad [6\text{–}40]$$

$$I = \frac{I_s}{\cos \beta S} \cos \beta s_r = I_{rs} \cos \beta s_r \qquad [6\text{–}41]$$

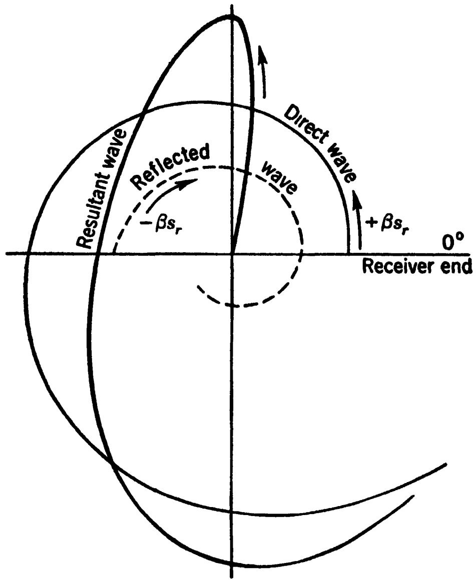


FIG. 6–11. Voltage loci for short-circuit line.

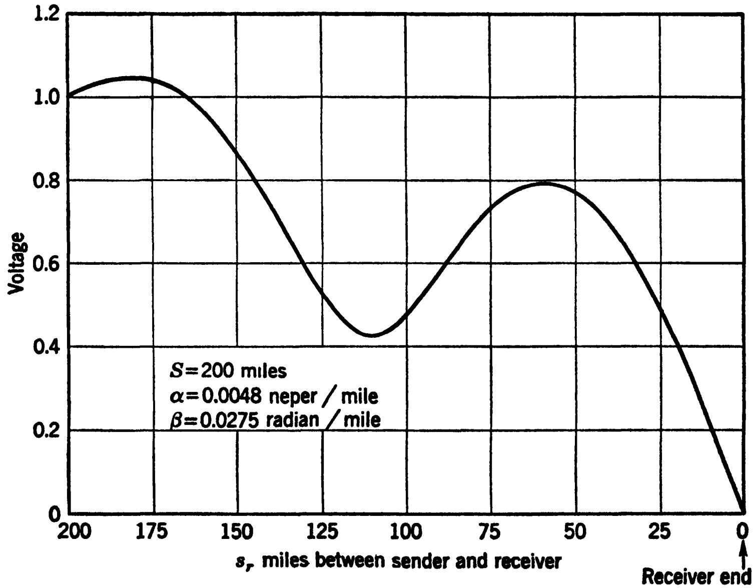


FIG. 6–12. Voltage along short-circuit line. (See Fig. 6–11.)

The curves of Fig. 6–13 show the open-circuit voltage and the short-circuit voltage on lines similar to those of Figs. 6–10 and 6–12 except that $\alpha = 0$. It will be seen, on comparing the curves with those of Figs. 6–10 and 6–12, that maxima and minima occur at corresponding points on the two sets of curves but that for finite attenuation the curves do not drop to zero at any point except at the receiving end of the short-circuit line.

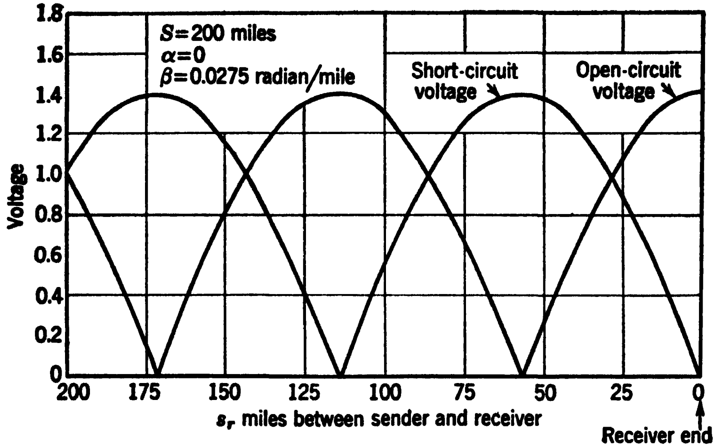


FIG. 6–13. Voltage on open- and short-circuit lines without attenuation.

52. The High-Frequency Line. If the frequency of the transmitted wave is high, certain simplifications can be made in the mathematical treatment of lines which lead to many useful and interesting results. Where resistance effects can be neglected, the significant operating characteristics of the high-frequency line can be brought more sharply into focus with the simplified equations which result from neglecting the losses.

If $R = 0$ and $G = 0$,

$$Z_0 = \sqrt{\frac{0 + j\omega L}{0 + j\omega C}} = \sqrt{\frac{L}{C}} \text{ ohms} \quad \text{[See equation 5–44]}$$

At high frequencies the internal inductance of a parallel-wire line is negligible, and the inductance is

$$L = 4 \ln \frac{d}{r} \times 10^{-7} \text{ henry/loop meter} \qquad [1\text{–}12]$$

and

$$C_{w\text{-to-}w} = \frac{1}{36 \times 10^9 \ln \frac{d}{r}} \text{ farad/meter} \qquad [1\text{–}30]$$

where ϵ_r is now taken as unity.

The characteristic impedance of the open-wire line then becomes

$$Z_0 = \sqrt{\frac{4 \ln \frac{d}{r} \times 10^{-7}}{\frac{1}{36 \times 10^9 \ln \frac{d}{r}}}} = 120 \ln \frac{d}{r} \doteq 276 \log \frac{d}{r} \text{ ohms}$$

$$= 120 \cosh^{-1} \frac{d}{2r} \qquad [6\text{–}42]$$

for $d \gg r$.

For coaxial conductors, under the same operating conditions,

$$L = 2 \ln \frac{b}{a} \times 10^{-7} \text{ henry/meter} \qquad [1\text{–}24]$$

$$C_{co} = \frac{1}{18 \times 10^9 \ln \frac{b}{a}} \text{ farad/meter} \qquad [1\text{–}35]$$

where ϵ_r is taken as unity, and

$$Z_0 = 60 \ln \frac{b}{a} \doteq 138 \log \frac{b}{a} \text{ ohms} \qquad [6\text{–}43]$$

From equation 5–32 it can be shown that

$$\beta = \frac{2\pi}{\lambda} = \omega\sqrt{LC} \qquad [6\text{–}44]$$

and hence

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = 3 \times 10^8 \text{ meters/sec} \qquad [6\text{–}45]$$

or the velocity of wave propagation on the lossless line is the same as the velocity of light.

The input impedance of the lossless line is obtained from equation 6–11 letting $a = \alpha S = 0$.

$$Z_s = Z_0 \frac{Z_r \cos \beta S + jZ_0 \sin \beta S}{Z_0 \cos \beta S + jZ_r \sin \beta S} \qquad [6\text{–}46]$$

For a short-circuit lossless line, $Z_r = 0$, and from equation 6–46 it is evident that

$$Z_{ss} = jZ_0 \tan \beta S = jZ_0 \tan 2\pi \frac{S}{\lambda} \text{ ohms} \qquad [6\text{–}47]$$

With $0 < S < \lambda/4$ or $0° < \beta S < 90°$, $\tan \beta S$ is positive, and hence the impedance is inductive reactance, whereas, with $\lambda/4 < S < \lambda/2$, $\tan \beta S$ is negative, and the reactance is capacitive. A graph of Z_{ss} versus S is shown in Fig. 6–14 and illustrates the alternate variation from inductance to capacitance of a short-circuit line as the length of the line increases. It is to be noted that the input impedance of the short-

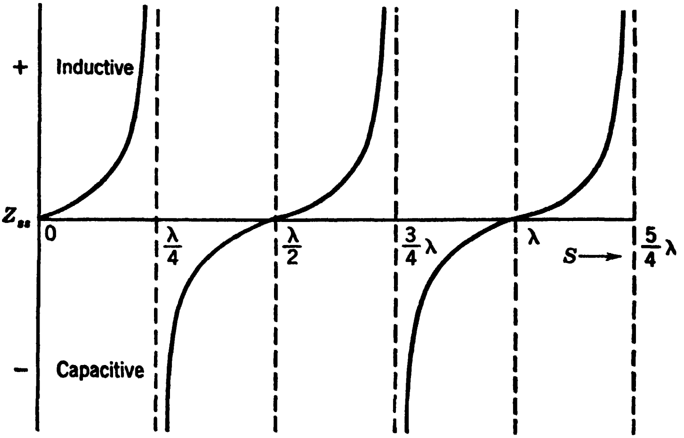


FIG. 6–14. Variation of Z_{ss} with wavelength on short-circuit lossless line.

circuit lossless line is infinite at the quarter-wavelength points, $S = \lambda/4$, $S = \frac{3}{4}\lambda$, etc., and zero at the half-wavelength points, $S = \lambda/2$, $S = \lambda$, etc. Also, at $S = \lambda/8$ this line has an inductive reactance equal in magnitude to its characteristic impedance.

For an open-circuit lossless line, $Z_r = \infty$, and, if both numerator and denominator of equation 6–46 are divided by Z_r and the limit of the resulting expression taken as $Z_r \to \infty$, it is evident that

$$Z_{so} = -jZ_0 \cot \beta S = -jZ_0 \cot 2\pi \frac{S}{\lambda} \text{ ohms} \qquad [6\text{–}48]$$

With $0 < S < \lambda/4$ or $0° < \beta S < 90°$, $\cot \beta S$ is positive, the impedance is negative reactance and hence capacitive, whereas with $\lambda/4 < S < \lambda/2$, $\cot \beta S$ is negative, the impedance is positive reactance and hence inductive. A graph of Z_{so} versus S is shown in Fig. 6–15 and

illustrates the alternate variation from capacitance to inductance of an open-circuit line as the length of the line increases. It is to be noted that the input impedance of the open-circuit lossless line is zero at the quarter-wavelength points and infinite at the half-wavelength points. Also, at $S = \lambda/8$ this line has a capacitive reactance equal in magnitude to its characteristic impedance.

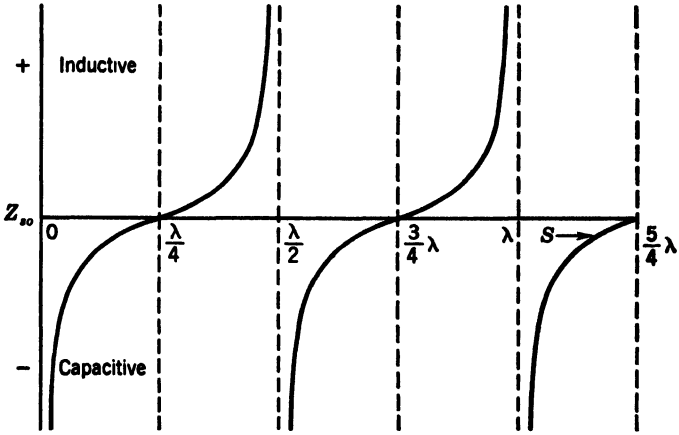


FIG. 6–15. Variation of Z_{so} with wavelength on open-circuit lossless line.

If the losses of the line are to be included in the analysis, the input impedance of the short-circuit line, which is given by equation 6–17, is

$$Z_{ss} = Z_0 \frac{\sqrt{\sinh^2 a + \sin^2 b} \;\Big/ \tan^{-1} \dfrac{\sin b \cosh a}{\cos b \sinh a}}{\sqrt{\sinh^2 a + \cos^2 b} \;\Big/ \tan^{-1} \dfrac{\sin b \sinh a}{\cos b \cosh a}} \qquad [6\text{–}17]$$

and reduces to the following whenever the line is of such a length that b is an odd multiple of $\pi/2$:

$$Z_{ss} = Z_0 \frac{\sqrt{\sinh^2 a + 1}\,/90^\circ}{\sinh a\,/90^\circ} \qquad [6\text{–}49]$$

$$= Z_0 \frac{\cosh a}{\sinh a} = \frac{Z_0}{\tanh a} \qquad [6\text{–}50]$$

For short lengths of line a is very small and $\tanh a \doteq a$, then

$$Z_{ss} \doteq \frac{Z_0}{a} = \frac{Z_0}{\alpha S}$$

Employing equation 5–46 and the fact that S can be written as $n\lambda/4$, we have

$$Z_{ss} \doteq \frac{2Z_0^2}{RS} = \frac{8Z_0^2}{n\lambda R} \text{ ohms} \qquad [6\text{–}51]$$

where n is the number of odd one-quarter wavelengths in the length of line.

When b is zero or even multiples of $\pi/2$ (integral multiples of half-wavelengths),

$$Z_{ss} = Z_0 \frac{\sinh a \underline{/0^\circ}}{\sqrt{\sinh^2 a + 1} \underline{/0^\circ}} \qquad [6\text{–}52]$$

$$= Z_0 \frac{\sinh a}{\cosh a} = Z_0 \tanh a \qquad [6\text{–}53]$$

and for short lengths of line, since $\tanh a \doteq a = \alpha S$

$$Z_{ss} \doteq Z_0 \frac{RS}{2Z_0} = \frac{RS}{2}$$

Writing S as $n'\lambda/2$ gives

$$Z_{ss} \doteq \frac{Rn'\lambda}{4} \text{ ohms} \qquad [6\text{–}54]$$

where n' is the number of one-half wavelengths in the line.

The input impedance of the open-circuit line, which is given by equation 6–14, is

$$Z_{so} = Z_0 \frac{\sqrt{\sinh^2 a + \cos^2 b} \underline{/ \tan^{-1} \frac{\sin b \sinh a}{\cos b \cosh a}}}{\sqrt{\sinh^2 a + \sin^2 b} \underline{/ \tan^{-1} \frac{\sin b \cosh a}{\cos b \sinh a}}} \qquad [6\text{–}14]$$

and reduces to the same expression as given by equation 6–50 whenever the line is of such a length that b is zero or equal to some integral multiple of π. That is,

$$Z_{so} = Z_0 \frac{\sqrt{\sinh^2 a + 1} \underline{/0^\circ}}{\sinh a \underline{/0^\circ}} \qquad [6\text{–}55]$$

$$= Z_0 \frac{\cosh a}{\sinh a} = \frac{Z_0}{\tanh a} \qquad [6\text{–}50]$$

For short lengths of line $\tanh a \doteq a$, then

$$Z_{so} \doteq \frac{Z_0}{a} = \frac{Z_0}{\alpha S}$$

Again employing equation 5–46 and writing S in terms of $n'\lambda/2$,

$$Z_{so} \doteq \frac{2Z_0^2}{RS} = \frac{4Z_0^2}{n'\lambda R} \text{ ohms} \qquad [6\text{–}56]$$

where n' is the number of one-half wavelengths in the length of line.

The resistance of a hard-drawn copper coaxial cable, taking into consideration the skin effect, is

$$R = 42.1\sqrt{f}\left(\frac{1}{a} + \frac{1}{b}\right) \times 10^{-7} \text{ ohm/meter} \qquad [1\text{–}83]$$

where the radii a and b must be in centimeters and f is the frequency of the transmitted wave in cycles per second. For two parallel hard-drawn copper wires the equation is

$$R = \frac{84.2\sqrt{f} \times 10^{-7}}{r_{cm}} \text{ ohm/loop meter} \qquad [1\text{–}81]$$

where r_{cm} is the radius of the conductor in centimeters.

For the case of two parallel hard-drawn copper wires, and substituting R from equation 1–81 and $\lambda(= c/f)$ into equation 6–51, there is obtained

$$Z_{ss} = \frac{9.50Z_0^2 r_{cm}\sqrt{f} \times 10^5}{nc} \text{ ohms} \qquad [6\text{–}57]$$

where r_{cm} is in centimeters, c is the velocity of light in meters per second $(=3 \times 10^8)$, f is in cycles per second, and n is the number of odd multiples of one-quarter wavelengths in the line. The importance of this case lies in the fact that the impedance is proportional to $\sqrt{f}$. It is also interesting to note that the impedance is also inversely proportional to the number of quarter wavelengths. The optimum ratio of wire spacing to wire radius for maximum impedance has been shown[1] to be about 8.

At high frequencies the quarter-wavelength line is very effective as a voltage amplifier. Equation 6–19 gives for the receiver voltage

$$V_{ro} = \frac{V_s}{\cosh \sqrt{zy}\, S}$$

or

$$\frac{V_{ro}}{V_s} = \frac{1}{\sqrt{\sinh^2 a + \cos^2 b}} \qquad [6\text{–}58]$$

[1] "Resonant Lines in Radio Circuits," by F. E. Terman, *Elec. Engineering*, Vol. 53, pp. 1046–1053.

At one-quarter wavelength $\cos b = 0$. Thus equation 6–58 becomes

$$\frac{V_{ro}}{V_s} = \frac{1}{\sinh a}$$
$$\doteq \frac{1}{\alpha S}$$

since $\sinh a = \sinh \alpha S \doteq \alpha S$. Then

$$\frac{V_{ro}}{V_s} \doteq \frac{2Z_0}{SR} = \frac{8Z_0 f}{Rc}$$

and for parallel-wire lines

$$\frac{V_{ro}}{V_s} = \frac{9.50 Z_0 r_{cm} \sqrt{f} \times 10^5}{c} \qquad [6\text{–}59]$$

where r_{cm} is the radius of the conductor in centimeters. Thus the voltage step-up also increases as the increase in $\sqrt{f}$.

At very high frequencies such quarter- and half-wavelength lines are very short and find many uses in radio and general laboratory technique. As an illustration, the quarter-wavelength step-up line may be used instead of an input transformer to a vacuum tube.

53. Illustrative Example. Let the value of V_{ro}/V_s be determined from equation 6–59 for a parallel-wire line constructed of 104-mil-diameter hard-drawn copper conductors spaced 18 inches center to center when operating at a frequency of 10^6 cycles per second. In this case,

$$Z_0 \doteq \sqrt{\frac{L}{C}} \text{ ohms}$$

For this wire size, $r = 0.052$ in. $= 0.00132$ meter,

$$mr = \sqrt{\frac{\omega\mu}{\rho}}\, r = \sqrt{\frac{2\pi \times 10^6 \times 4\pi \times 10^{-7}}{1.77 \times 10^{-8}}} \times 0.00132 = 27.9$$

and L_{ac}/L_{dc} from Fig. 1–9 is 0.1. Then

$$L = \left(0.1 + 4 \ln \frac{d}{r}\right) \times 10^{-7} = 23.5 \times 10^{-7} \text{ henry/loop meter}$$

$$C = \frac{1}{36 \times 10^9 \ln \frac{d}{r}} = 4.75 \times 10^{-12} \text{ farad/meter}$$

and

$$Z_0 \doteq \sqrt{\frac{23.5 \times 10^{-7}}{4.75 \times 10^{-12}}} = 704 \text{ ohms}$$

From equation 6–59,

$$\frac{V_{ro}}{V_s} = \frac{9.50 \times 704 \times 0.132 \times \sqrt{10^6} \times 10^5}{3 \times 10^8} = 294$$

Thus the step-up ratio obtained by the use of the one-quarter-wavelength line is 294, or expressed in decibels

$$\text{db} = 20 \log 294 = 49.4$$

54. Distance to an Open Circuit on a Line. If an open circuit should develop on a line at an unknown distance from the sending end, it is desirable to have some method of determining this distance by making measurements at the sending end. The absolute value of the input impedance of an open line is given by

$$Z_{so} = Z_0 \frac{\sqrt{\sinh^2 a + \cos^2 b}}{\sqrt{\sinh^2 a + \sin^2 b}} \tag{6–60}$$

It has already been seen that Z_{so} is a function of the length of the line, since $b = \beta l$, and the value of the impedance goes through successive maxima at every half-cycle of b. Also b is a function of frequency. (See equations 5–32, 5–34, and 5–40.) Thus, as the frequency is increased, on a line of fixed length the value of b will change, thereby causing Z_{so} to go through a series of maxima and minima. Also, as the frequency increases, the wavelength decreases, so that the length of the line in terms of wavelength gradually increases. Let it be assumed that on an open line at a certain frequency, f_1, $b = n\pi$, where n is an integer, whence Z_{so} is a maximum. This line will be exactly $n/2$ wavelengths long. Since $b = \beta l$, and $\beta = 2\pi/\lambda_1$, then $b = 2\pi l/\lambda_1 = n\pi$, whence $l = n\lambda_1/2$. Therefore $n/2 = D/\lambda_1$ where D is the distance from the sending end to the fault, in miles. Suppose now that the frequency is slowly increased, b will progress in value from $n\pi$ to $(n\pi + \pi/2)$ whereupon Z_{so} will become a minimum. A further increase in frequency will increase b to $(n\pi + \pi)$ or $(n + 1)\pi$ which will again make $\cos^2 b = 1$, and a new maximum value of Z_{so} will result. For this new frequency f_2 the line will be exactly $(n + 1)/2$ wavelengths long and $(n + 1)/2 = D/\lambda_2$.

These two equations may be written

$$\frac{n}{2} = \frac{D}{\lambda_1} = \frac{Df_1}{v_1} \tag{6–61}$$

and

$$\frac{n}{2} + \frac{1}{2} = \frac{D}{\lambda_2} = \frac{Df_2}{v_2} \tag{6–62}$$

where the subscripts refer to the first and second frequency readings respectively. $n/2$ may be eliminated from these equations, giving

$$\frac{Df_1}{v_1} + \frac{1}{2} = \frac{Df_2}{v_2}$$

from which

$$D = \frac{1}{2}\left(\frac{v_1 v_2}{v_1 f_2 - v_2 f_1}\right) \qquad [6\text{–}63]$$

Two adjacent maxima, however, are usually close enough together that v_1 and v_2 may be considered equal; thus the equation may be written

$$D \doteq \frac{v}{2(f_2 - f_1)} \qquad [6\text{–}64]$$

In the above equations v is the velocity of propagation and can be calculated from $v = \omega/\beta$.

Equations 6–61 and 6–62 may also be solved for D in terms of wavelengths, in which case

$$D = \frac{1}{2} \cdot \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \qquad [6\text{–}65]$$

or in terms of β,

$$D = \frac{\pi}{\beta_2 - \beta_1} \qquad [6\text{–}66]$$

PROBLEMS

Make all calculations at 796 cycles per second in Probs. 1 to 13 inclusive.

6–1. Calculate the sending-end impedances for line A of Prob. 5–1 when terminated in (a) a pure resistance $R = Z_0$, (b) a pure resistance $R = Z_0/2$, and (c) a pure resistance $R = 2Z_0$. $Z_0 = 619\underline{/-3.53^\circ}$ ohms.

6–2. Plot the open- and short-circuit impedances of lines having lengths varying from 0 to 2λ which have parameters as given for the line A of Prob. 5–1.

6–3. Plot V_{ro} for lines having lengths varying from 0 to λ which have parameters as given for the line A of Prob. 5–1. Over what lengths of line is $V_{ro} > V_s$? Let $V_s = 1$ volt.

6–4. Plot V_{ro} for lines having lengths varying from 0 to λ in which $\alpha = 0.0100$ neper/mile and $\beta = 0.0275$ radian/mile. Let $V_s = 1$ volt.

6–5. Plot V_{ro} for lines having lengths varying from 0 to λ in which $\alpha = 0.0200$ neper/mile and $\beta = 0.0275$ radian/mile. Let $V_s = 1$ volt.

6–6. Plot V_{ro} for lines having lengths varying from 0 to λ in which $\alpha = 0.0275$ neper/mile and $\beta = 0.0275$ radian/mile. Let $V_s = 1$ volt.

6–7. Calculate the short-circuit current I_{rs} for the 200-mile line A of Prob. 5–1 when $V_s = 1$ volt.

6-8. Plot the voltage variation along the line A of Prob. 5-1 when the line is open-circuited. $V_s = 1$ volt. Compare results with those of Prob. 5-2.

Plot the voltage variation along a 55-mile line having the parameters per loop mile of line A of Prob. 5-1 when the line is open-circuited. $V_s = 1$ volt.

Plot the voltage variation along a 145.2-mile line having the parameters per loop mile of line A of Prob. 5-1 when the line is open-circuited. $V_s = 1$ volt.

6-9. Plot the current variation along the line A of Prob. 5-1 when the line is open-circuited. $V_s = 1$ volt.

6-10. Plot the impedance variation along the line A of Prob. 5-1 when the line is open-circuited. Compare results with Prob. 6-2.

6-11. Plot the voltage variation along the line A of Prob. 5-1 when the line is short-circuited. $V_s = 1$ volt.

6-12. Plot the current variation along the line A of Prob. 5-1 when the line is short-circuited. $V_s = 1$ volt.

6-13. Plot the impedance variation along the line A of Prob. 5-1 when the line is short-circuited. Compare results with Probs. 6-2 and 6-10.

6-14. An antenna feeder is 27.5 meters long and is made up of two parallel wires ⅛ inch in diameter and spaced 6 inches center to center. It supplies power at 5 megacycles per second. At a point 7.5 meters from the receiver end a short-circuit

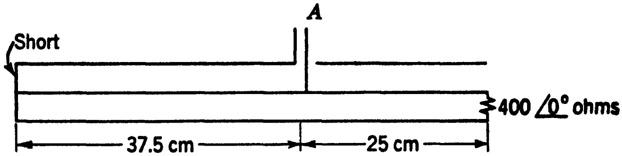


FIG. 6-16. For use in connection with Prob. 6-15.

stub of 7.5 meters length is attached. What is the input impedance of this line when the line is open at the receiver terminals? Both the line and the stub have the same $Z_0(= \sqrt{L/C})$.

6-15. What is the impedance looking into the coaxial line of Fig. 6-16 at point A? Power is supplied at 200 megacycles per second to a load of $400\underline{/0^\circ}$ ohms. The b/a ratio for the coaxial conductors is 3.22.

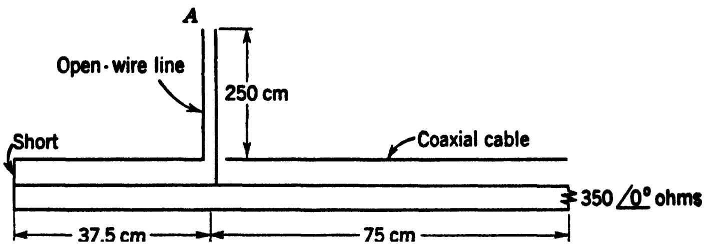


FIG. 6-17. For use in connection with Prob. 6-16.

6-16. What is the impedance looking into the open-wire line of Fig. 6-17 at point A? The open-wire line feeds into the coaxial cable which is terminated in an

resistance. The characteristic impedance of the open-wire line is $350\underline{/0°}$ ohms. Power is supplied at 200 megacycles per second to the load of $350\underline{/0°}$ ohms. The b/a ratio for the coaxial conductors is 3.22.

6-17. Given a section of parallel-wire line which has a short circuit at one end and a capacitance C at the other as shown in Fig. 6-18. The wire is ⅛ inch in diameter,

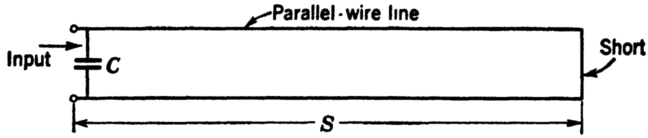


Fig. 6-18. For use in connection with Prob. 6-17.

and the spacing is 6 inches center to center. Power is supplied at 100 megacycles per second, and $C = 8$ $\mu\mu$f. Find the length S so that the circuit will present infinite impedance at the input. Neglect the resistance of the line.

6-18. The coaxial cable of Fig. 6-19 is one-quarter wavelength long and is terminated in a capacitance of 10 micromicrofarads. Assume that $\alpha = 0$. What voltage

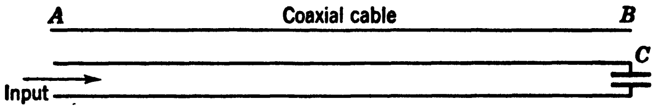


Fig. 6-19. For use in connection with Prob. 6-18.

must be applied at A in order that 1.0 volt will appear across the capacitance at B? Let the ratio b/a be 2.5 and $f = 6$ mc/sec.

6-19. (*a*) Find the input impedance of a quarter-wavelength short-circuited coaxial line operating at 200 megacycles per second if $a = 0.1$ cm and $b = 0.322$ cm. The conductor material is hard-drawn copper.

(*b*) Find the ratio of inductive reactance to resistance for the quarter-wavelength line, basing the calculation of inductance on equation 1-24.

6-20. A parallel-wire line is composed of hard-drawn copper conductors 0.5 centimeter in radius and separated center to center by a distance of 4 centimeters.

(*a*) Find the impedance looking into a quarter wavelength of short-circuited line at a frequency of 200 megacycles per second.

(*b*) Find the ratio of inductive reactance to resistance of the quarter-wavelength line, basing the inductance calculation on equation 1-12.

6-21. Given a parallel-wire line constructed of No. 10 AWG hard-drawn copper wires spaced 10 inches center to center. The length of the line is $\lambda/4$. The input voltage V_s is 1.0 volt, and the frequency to be transmitted is 10^6 cycles per second. Find V_{ro}.

6-22. Given a coaxial cable operating at 10^8 cycles per second. The radii are 0.3 and 2.0 centimeters, and the resistance per unit length is given by equation 1-83. The conductor material is hard-drawn copper. Find:

(*a*) The voltage step-up for one-quarter wavelength of open-circuit line.

(*b*) The input impedance Z_{so} for one-quarter wavelength of open-circuit line.

(*c*) The input impedance for one-half wavelength of open-circuit line.

6-23. A coaxial line operates at 10^6 cycles per second. The radii are $a = 1$ cm and $b = 2$ cm. (a) What resistance per meter must this line have in order to just prevent Ferranti effect from occurring if the length of the line is 0.005λ? (b) What resistance per meter must this line have in order to just prevent Ferranti effect from occurring if the length of the line is 0.25λ?

6-24. A line is known to have the following parameters per loop mile:

$$C = 0.01\ \mu\text{f}$$

$$L = 0.005 \text{ henry}$$

$$G \doteq 0$$

$$\begin{aligned} R &= 6 \text{ ohms at } f = 1591 \text{ cycles/sec} \\ &= 7 \text{ ohms at } f = 1860 \text{ cycles/sec} \\ &= 8 \text{ ohms at } f = 2130 \text{ cycles/sec} \end{aligned}$$

Calculate Z_0 for each of the specified frequencies, namely, 1591, 1860, and 2130 cycles per second.

6-25. Calculate the attenuation and phase shift for a 133-mile length of the line specified in Prob. 6-24 at each of the three frequencies.

Calculate the magnitude of Z_{so} for each frequency.

6-26. A line is known to have the following parameters per loop mile:

$$R = 10.4 \text{ ohms} \qquad L = 0.00367 \text{ henry}$$

$$C = 0.00835\ \mu\text{f} \qquad G = 0$$

An open circuit developed on this line at an unknown distance X from the sending end. It is found on making measurements of Z_{so} at the sending end that two successive maxima of Z_{so} occur at frequencies of 1360 and 1820 cycles per second. Find the distance X.

CHAPTER VII

REFLECTION LOSSES

In the foregoing chapters the equations for the calculation of current and voltage on the general line have been derived. For a concluding section on such lines it is desirable to treat in more detail the problem of determining the line losses for lines which are terminated in any manner whatever. It is true of course that the current and voltage can be calculated at the line input and output terminals and the power transmitted determined therefrom. However, a clearer understanding of the effects of mismatched impedances is desirable. The present chapter takes up the concepts of "reflection factor" and "insertion loss" and also presents miscellaneous material concerning measurements on lines.

55. Reflection Factor. The general equations presented in the preceding chapter indicate clearly that considerable disturbance takes place on a line which is not properly terminated. In Chapter III it was shown, by means of the maximum power theorem, that in order for maximum power to be transmitted the load impedance should equal the conjugate of the generator impedance. This relation can be applied anywhere in a circuit by means of Thévenin's theorem; and from it one is led to the conclusion that, if a line is to be the best available for transmission of power, impedances measured both ways at any given point along the line should be conjugates of one another. In the treatment of transmission lines, however, where the termination should be Z_0 in order to prevent reflections, it is generally advisable to base the argument not on the result of the maximum power theorem but on a condition where the two impedances involved are equal both in magnitude and in angle. This condition of equality prevents reflections, as stated in Chapter V, and accordingly prevents echoes returning from the receiving end of the line which would interfere with new signals being transmitted from the sending end. The necessary improvement in quality gained by making the terminating impedance *equal* to Z_0 is thus the determining factor, and the conjugate matching for maximum power transfer must be abandoned. This condition of equal impedances is what is meant when it is said that two impedances are *matched*. As an illustration, the optimum condition is taken to be represented by

a generator of impedance Z_g working into an impedance Z_g and not into the conjugate of Z_g. Thus for an ideal transmission line the generator of impedance Z_g should work into a line whose characteristic impedance Z_0 is equal to Z_g and whose terminating impedance is Z_0. If the generator and terminating impedances differ from Z_0, additional losses in transmission will exist beyond the $\epsilon^{-\alpha l}$. In the general case of no matching there is an additional loss where the generator connects to the line and also a loss where the line connects to the terminating impedance. If all of these individual losses could be separately determined, then there would exist a means of calculating the total loss due to all of the mismatching. This problem will be treated in two ways, the first on the basis of a general discussion and the second on the basis of the exact equations previously derived.

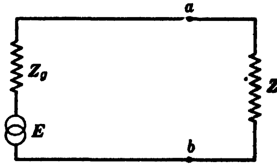


FIG. 7-1. Mismatched impedances.

It is first necessary to determine the additional loss occasioned by mismatched impedances such as shown in the simple circuit of Fig. 7-1. The current delivered is given by the equation,

$$I_1 = \frac{E}{Z_g + Z_r} \tag{7-1}$$

Now let it be assumed that the circuit is opened at a–b and an ideal transformer inserted whose impedance ratio is Z_g/Z_r. This transformer presents to the generator an impedance Z_g and to the load an impedance Z_r, so that all impedances are matched and reflections are eliminated. In the ideal transformer $I_p/I_s = N_s/N_p$, and from elementary transformer theory $Z_g/Z_r = (N_p/N_s)^2$. Thus $I_p/I_s = \sqrt{Z_r/Z_g}$ or $I_s = I_p\sqrt{Z_g/Z_r}$. The current I_p, in these matched impedances, will be given as $I_1 = E/(2Z_g)$ from equation 7-1, and

$$I_2 = I_s = \frac{E}{2Z_g}\sqrt{\frac{Z_g}{Z_r}} = \frac{E}{2\sqrt{Z_gZ_r}} \tag{7-2}$$

Since it is the mismatched case as compared to the matched case that is desired, form the ratio,

$$\frac{I_1}{I_2} = \frac{E}{|Z_g + Z_r|}\cdot\frac{2\sqrt{Z_gZ_r}}{E} = \frac{2\sqrt{Z_gZ_r}}{|Z_g + Z_r|}$$

This current ratio which indicates the deviation from the optimum

condition is called the *reflection factor* and is written

$$k = \frac{2\sqrt{Z_g Z_r}}{|Z_g + Z_r|} \tag{7-3}$$

The number of nepers corresponding to this ratio is

$$N = \ln k = \ln \frac{2\sqrt{Z_g Z_r}}{|Z_g + Z_r|} \tag{7-4}$$

If it is desired to express this quantity in decibels, as is often the case, it may be written as follows, remembering that here the number of decibels merely refers to the current ratio and has nothing to do with power or voltage ratios,

$$\text{db} = 20 \log \frac{2\sqrt{Z_g Z_r}}{|Z_g + Z_r|} \tag{7-5}$$

If k of equation 7–3 is greater than unity, then the reflection loss becomes a reflection gain, and the loss given by equations 7–4 and 7–5 would appear as a positive quantity. In this material *reflection loss* and *reflection gain* refer to a decrease or increase respectively in the ratio of output to input *current.* From what has been said it could be inferred in general that wherever, in a line, a mismatch of impedances occurs, a loss given by equation 7–4 would take place. Thus, if a line whose characteristic impedance is Z_0 is connected between the generator and load of Fig. 7–1, two mismatches occur, and three losses are present as follows:

(1) At the mismatch between generator and line where

$$k_1 = \frac{2\sqrt{Z_g Z_0}}{|Z_g + Z_0|}$$

(2) In the line itself,

$$\epsilon^{-\alpha l}$$

(3) At the mismatch between the line and the terminating impedance Z_r, where

$$k_2 = \frac{2\sqrt{Z_0 Z_r}}{|Z_0 + Z_r|}$$

The foregoing material requires careful consideration. In the first place the input impedance to the line at the generator was taken as Z_0 instead of Z_s as it appears in equation 6–11, and the line impedance measured back into the line at the receiver end has been taken as Z_0 in place of the right-hand side of equation 6–11 wherein Z_g replaces Z_r.

This introduces an approximation which must be considered later. Second, the above three losses exist in addition to the loss presented by *an ideal case of no line and perfect matching of impedances.* Thus the total product of the three losses ($\epsilon^{-\alpha l}k_1k_2$) represents a loss in addition to that of the ideal condition of Fig. 7–1 with $Z_g = Z_r$. The number of nepers loss above the loss incurred if the generator were perfectly matched is

$$N' = -\alpha l + \ln k_1 + \ln k_2$$

If there exists a generator with an impedance Z_g and a load of impedance Z_r which must be used, and if it is required to find the loss which will result when the line of impedance Z_0 and attenuation factor $\epsilon^{-\alpha l}$ is placed between the generator and load, all the losses listed will occur. However, since the best possible condition takes place when Z_r is directly connected to the generator, all the above loss cannot be charged to the insertion of the line, because with Z_r and the generator connected some loss above the optimum condition referred to already exists. As a matter of fact this " best " condition already involves a loss given by

$$k_3 = \frac{2\sqrt{Z_gZ_r}}{|Z_g + Z_r|}$$

which must be applied as a correction to the above ratio. A ratio of

$$\frac{\epsilon^{-\alpha l}k_1k_2}{k_3}$$

represents the actual loss caused by the insertion of the line. Written out in nepers, this is

$$N' = -\alpha l + \ln k_1 + \ln k_2 - \ln k_3$$

When written in terms of a ratio greater than unity, the ratio is called *insertion loss,* and

$$N = \alpha l + \ln \frac{1}{k_1} + \ln \frac{1}{k_2} - \ln \frac{1}{k_3} \qquad [7\text{–}6]$$

The term $\ln 1/k$ is called *reflection loss.* The definition of insertion loss may now be given as " . . . the loss which is caused by inserting a line between a generator and a load."

56. Insertion Loss. In the preceding article insertion loss was given by equation 7–6 where the line loss and the effects of three possible mismatches were taken into consideration. The equation was not proved but was merely written down on the basis of the definition of

reflection factor and a knowledge of line loss. In this article it will be proved that equation 7–6 is correct if the line attenuation is sufficiently high, and it is to be remembered that *insertion loss* refers as before to a decrease in current ratio. An approximation arises due to the preceding assumption that Z_0 is the input impedance of the line. Refer to Fig. 7–2. The receiving-end current without the line is

$$I_1 = \frac{E}{Z_g + Z_r}$$

With the line in place the sending-end current is

$$I_s = \frac{E}{Z_g + Z_s}$$

and the receiving-end current is given by equation 6–5*a* as

$$\begin{aligned} I_2 &= \frac{I_s}{\cosh \gamma l + \dfrac{Z_r}{Z_0} \sinh \gamma l} \\ &= \frac{E}{(Z_g + Z_s)\left(\cosh \gamma l + \dfrac{Z_r}{Z_0} \sinh \gamma l\right)} \end{aligned}$$

It is desired that the current ratio be larger than unity so

$$\begin{aligned} \frac{I_1}{I_2} &= \frac{E}{(Z_g + Z_r)} \cdot \frac{(Z_g + Z_s)\left(\cosh \gamma l + \dfrac{Z_r}{Z_0} \sinh \gamma l\right)}{E} \\ &= \left(\cosh \gamma l + \frac{Z_r}{Z_0} \sinh \gamma l\right) \cdot \frac{Z_g + Z_0 \left[\dfrac{Z_r \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_r \sinh \gamma l}\right]}{Z_g + Z_r} \\ &= \frac{\left(\cosh \gamma l + \dfrac{Z_r}{Z_0} \sinh \gamma l\right) Z_g + Z_r \cosh \gamma l + Z_0 \sinh \gamma l}{Z_g + Z_r} \\ &= \frac{(Z_g + Z_r) \cosh \gamma l + \left(\dfrac{Z_g Z_r}{Z_0} + Z_0\right) \sinh \gamma l}{Z_g + Z_r} \\ &= \cosh \gamma l + \frac{Z_g Z_r + Z_0^2}{Z_0(Z_g + Z_r)} \sinh \gamma l \end{aligned} \qquad [7\text{–}7]$$

Equation 7–7 can be changed into an expression in exponentials. Let the coefficient of the sinh term be represented by A. Then

$$\frac{I_1}{I_2} = \frac{\epsilon^a \epsilon^{jb} + \epsilon^{-a}\epsilon^{-jb}}{2} + A\frac{\epsilon^a\epsilon^{jb} - \epsilon^{-a}\epsilon^{-jb}}{2}$$

Collecting the coefficients of ϵ^a and ϵ^{-a}

$$\frac{I_1}{I_2} = \frac{\epsilon^a}{2}(1 + A)\epsilon^{jb} + \frac{\epsilon^{-a}}{2}(1 - A)\epsilon^{-jb} \qquad [7\text{–}8]$$

In this equation it is seen that the current ratio is made up of a part which continually increases with line length, through the term ϵ^a, and a part which decreases with line length on account of ϵ^{-a}. The terms

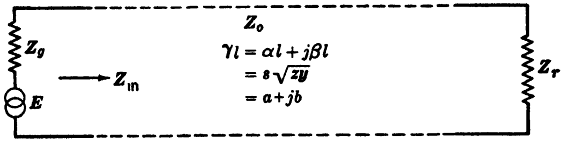


FIG. 7–2. Mismatched transmission line.

$(1 + A)$, $(1 - A)$, ϵ^{jb} and ϵ^{-jb} are complex. If the line length is sufficiently great, the second term on the right of equation 7–8 can be neglected. It is then necessary to find

$$\frac{I_1}{I_2} = \epsilon^a \frac{|1 + A|}{2}|\epsilon^{jb}| \qquad [7\text{–}9]$$

in order to determine the ratio representing the decrease in current due to the insertion of the line. Note that $|\epsilon^{jb}| = |\cos b + j\sin b| = 1$. $|1 + A|/2$ can be found by expansion in terms of the Z's.

$$\frac{1+A}{2} = \frac{1}{2} + \frac{Z_gZ_r + Z_0^2}{2Z_0(Z_g + Z_r)}$$

$$= \frac{Z_0Z_g + Z_0Z_r + Z_gZ_r + Z_0^2}{2Z_0(Z_g + Z_r)}$$

$$= \frac{(Z_0 + Z_r)(Z_g + Z_0)}{2Z_0(Z_g + Z_r)}$$

Let the numerator and denominator of the right-hand side be multiplied by $2\sqrt{Z_gZ_r}$. There results

$$\frac{1+A}{2} = \frac{(Z_0 + Z_r)(Z_g + Z_0)2\sqrt{Z_gZ_r}}{4Z_0\sqrt{Z_gZ_r}(Z_g + Z_r)}$$

This can be rearranged as follows and is expressed as an absolute value.

$$\frac{|1 + A|}{2} = \frac{|Z_g + Z_0|}{2\sqrt{Z_g Z_0}} \cdot \frac{|Z_0 + Z_r|}{2\sqrt{Z_0 Z_r}} \cdot \frac{2\sqrt{Z_g Z_r}}{|Z_g + Z_r|} \tag{7-10}$$

Let this be substituted into equation 7–9.

$$\frac{I_1}{I_2} = \epsilon^a \frac{|Z_g + Z_0|}{2\sqrt{Z_g Z_0}} \cdot \frac{|Z_0 + Z_r|}{2\sqrt{Z_0 Z_r}} \cdot \frac{2\sqrt{Z_g Z_r}}{|Z_g + Z_r|} \tag{7-11}$$

By taking the logarithm of both sides of this equation, remembering that $a = \alpha l$, and referring to the definitions of k_1, k_2, and k_3, equation 7–6 is obtained.

This development shows that the only assumption employed in setting up equation 7–6 is that the line is considered to be very long. As mentioned above, this indeed is the assumption made when Z_0 was used in place of the actual impedance of the line.

There are conditions where the length of the line will not affect the validity of equation 7–11. If either $Z_g = Z_0$, or $Z_r = Z_0$, then this equation is exact. This can be seen by reference to equation 7–8 where, if $A = 1$, it is seen that the ϵ^{-a} term disappears. To see what this means in the relations of Z_g, Z_0, and Z_r expand $(1 - A)$. From equation 7–7

$$\begin{aligned} 1 - A &= \frac{Z_0 Z_g + Z_0 Z_r - Z_g Z_r - Z_0^2}{Z_0(Z_g + Z_r)} \\ &= \frac{Z_g(Z_0 - Z_r) - Z_0(Z_0 - Z_r)}{Z_0(Z_g + Z_r)} \\ &= \frac{(Z_g - Z_0)(Z_0 - Z_r)}{Z_0(Z_g + Z_r)} \end{aligned} \tag{7-12}$$

Thus $(1 - A) = 0$ if either $Z_g = Z_0$ or $Z_r = Z_0$.

57. Illustrative Example. In order to present the ideas concerning insertion loss more adequately, a typical example will be worked out on the basis of both the approximate and exact methods. The insertion loss produced by inserting 200 miles of the unloaded line of Art. 33 between a generator of 200 ohms resistance and a load of 400 ohms resistance will be determined. For the unloaded line

$Z_g = 200$ ohms $\qquad Z_0 = 745\underline{/-13.45°} = 725 - j173.4$ ohms

$Z_r = 400$ ohms $\qquad \alpha = 0.00712$ neper/mile

$S = 200$ miles $\qquad \beta = 0.0288$ radian/mile

Equation 7–6 may be applied directly. In this case $\epsilon^{-a} = \epsilon^{-\alpha l} = \epsilon^{-0.00712 \times 200} = \epsilon^{-1.424} = 0.241$. The value of $(1 - A)$ also affects this approximation so the value of ϵ^{-a} cannot be used as a definite indication of the satisfactoriness of the result. Returning to equation 7–6, $\alpha l = 1.424$ nepers. The k's are given as follows:

$$k_1 = \frac{2\sqrt{Z_g Z_0}}{|Z_g + Z_0|} = \frac{2\sqrt{200 \times 745}}{|925 - j173.4|} = 0.821$$

$$k_2 = \frac{2\sqrt{Z_0 Z_r}}{|Z_0 + Z_r|} = \frac{2\sqrt{745 \times 400}}{|1125 - j173.4|} = 0.960$$

$$k_3 = \frac{2\sqrt{Z_g Z_r}}{|Z_g + Z_r|} = \frac{2\sqrt{200 \times 400}}{600} = 0.943$$

Equation 7–6 becomes

$$\begin{aligned} N &= 1.424 + \ln 1.219 + \ln 1.041 - \ln 1.061 \\ &= 1.424 + 0.198 + 0.0402 - 0.0593 = \mathbf{1.603} \text{ neper} \\ &= 13.92 \text{ db} \end{aligned}$$

In order to find the exact insertion loss it is necessary to find the ratio of I_1 to I_2 from equation 7–7. This will represent the change in load current due to the line being placed between the generator and the load.

$$\frac{I_1}{I_2} = \cosh \gamma l + \frac{Z_g Z_r + Z_0^2}{Z_0(Z_g + Z_r)} \sinh \gamma l$$

$$\begin{aligned} \gamma l = (\alpha + j\beta)S &= (0.00712 + j0.0288) \times 200 \\ &= 1.424 + j5.76 = a + jb \end{aligned}$$

$$\begin{aligned} \cosh \gamma l &= \sqrt{\sinh^2 a + \cos^2 b} \,\underline{/\tan^{-1} \frac{\sin b \sinh a}{\cos b \cosh a}} \\ &= \sqrt{\sinh^2 1.424 + \cos^2 330.0^\circ} \,\underline{/\tan^{-1} \frac{-0.50 \times 1.957}{0.866 \times 2.197}} \\ &= \sqrt{(1.957)^2 + (0.866)^2} \,\underline{/\tan^{-1} - 0.514} \\ &= 2.140 \underline{/-27.2^\circ} = 1.903 - j0.977 \end{aligned}$$

$$\begin{aligned} \sinh \gamma l &= \sqrt{\sinh^2 a + \sin^2 b} \,\underline{/\tan^{-1} \frac{\sin b \cosh a}{\cos b \sinh a}} \\ &= \sqrt{3.830 + 0.250} \,\underline{/\tan^{-1} - 0.648} \\ &= 2.020 \underline{/-32.95^\circ} \end{aligned}$$

$$\frac{Z_gZ_r + Z_0^2}{Z_0(Z_g + Z_r)} = \frac{200 \times 400 + (745\angle -13.45^\circ)^2}{745\angle -13.45^\circ \times 600}$$

$$= \frac{80{,}000 + 555{,}000\angle -26.90^\circ}{447{,}000\angle -13.45^\circ}$$

$$= \frac{80{,}000 + 495{,}000 - j251{,}000}{447{,}000\angle -13.45^\circ}$$

$$= \frac{627{,}000\angle -23.6^\circ}{447{,}000\angle -13.45^\circ} = 1.402\angle -10.15^\circ$$

Thus

$$I_1/I_2 = 1.903 - j0.977 + 1.402\angle -10.15^\circ \times 2.020\angle -32.95^\circ$$

$$= 1.903 - j0.977 + 2.832\angle -43.10^\circ$$

$$= 1.903 - j0.977 + 2.067 - j1.934$$

$$= 3.97 - j2.91 = 4.92\angle -36.24^\circ$$

or $I_1/I_2 = 4.92$ and the nepers loss is $\ln 4.92 = 1.593$. In decibels this is 13.84.

A comparison of the two methods shows that in this case the approximation leads to a loss of 13.92 decibels, compared with the correct value of 13.84 decibels.

58. Determination of Equivalent T Section. It has been shown that a network can be represented, at a given frequency, by a T section (or π section). It is now proposed to derive the equations which will give the elements of such a T section if the values of Z_0 and γ for a line with uniformly distributed constants are known. Both the equivalent T section and the line are to have the same Z_0, Z_{so}, and Z_{ss}.

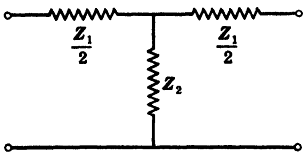


FIG. 7-3. T section.

For the T section of Fig. 7-3, using the notation employed in Chapter IV, the open-circuit impedance is

$$Z_{so} = \frac{Z_1}{2} + Z_2$$

and from equation 6-14

$$Z_{so} = Z_0 \coth \gamma l$$

Equating the two expressions for Z_{so} gives

$$Z_0 \coth \gamma l = \frac{Z_1}{2} + Z_2$$

from which

$$Z_1 = 2Z_0 \coth \gamma l - 2Z_2'$$

Substituting this value of Z_1 into the expression for Z_0^2, obtained from equation 4–4, namely,

$$Z_0^2 = Z_1 Z_2 + \frac{Z_1^2}{4}$$

gives

$$Z_0^2 = 2Z_0 Z_2 \coth \gamma l - 2Z_2^2 + Z_0^2 \coth^2 \gamma l - 2Z_0 Z_2 \coth \gamma l + Z_2^2$$

or

$$Z_0^2 = Z_0^2 \coth^2 \gamma l - Z_2^2$$

Then

$$Z_2 = Z_0 \sqrt{\coth^2 \gamma l - 1} = \frac{Z_0}{\sinh \gamma l} \qquad [7\text{–}13]$$

and

$$Z_1 = 2Z_0 \coth \gamma l - \frac{2Z_0}{\sinh \gamma l} = 2Z_0 \frac{\cosh \gamma l - 1}{\sinh \gamma l} \qquad [7\text{–}14]$$

Equations 7–13 and 7–14 give the values for the equivalent T section of Fig. 7–3.

If γl is sufficiently small, $\cosh \gamma l \doteq 1 + \frac{(\gamma l)^2}{2!}$ and $\sinh \gamma l \doteq \gamma l$, and these equations reduce to the approximate forms,

$$Z_1 = 2Z_0 \left(\frac{\gamma l}{2}\right) = Z_0 \gamma l = \sqrt{\frac{z}{y}} \sqrt{zy}\, l = zl \qquad [7\text{–}15]$$

and

$$Z_2 = \frac{Z_0}{\gamma l} = \frac{\sqrt{\frac{z}{y}}}{\sqrt{zy}\, l} = \frac{1}{yl} \qquad [7\text{–}16]$$

59. Determination of Line Constants from Direct Measurements. In the previous article it was shown how the elements of a T section, equivalent to a given length of line, can be found. Thus, given a line for which γ and Z_0 are known, the problem was to find the equivalent T-section elements. Another type of problem which arises is that of finding Z_0, γ, R, L, G, and C by making measurements at the end of a line, the length of which alone is known. The simplest measurements which can be made are those which yield the input impedances for the open- and short-circuit conditions. From these open- and

short-circuit impedances the values of γ and Z_0 can be found, and these latter values used to determine the elementary constants.

Begin with the two equations

$$Z_0 = \sqrt{\frac{z}{y}} \tag{5-15}$$

and

$$\gamma = \sqrt{zy} \tag{7-17}$$

Using equations 6–14 and 6–16, namely,

$$Z_{so} = Z_0 \coth \sqrt{zy}\, S \tag{6-14}$$

$$Z_{ss} = Z_0 \tanh \sqrt{zy}\, S \tag{6-16}$$

equation 5–15 becomes

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{Z_{ss}Z_{so}} \tag{7-18}$$

and from these same equations it is seen that

$$\tanh \sqrt{zy}\, S = \tanh \gamma l = \sqrt{\frac{Z_{ss}}{Z_{so}}} \tag{7-19}$$

From equations 5–15 and 7–17,

$$\gamma Z_0 = z = R + j\omega L \tag{7-20}$$

and

$$\frac{\gamma}{Z_0} = y = G + j\omega C \tag{7-21}$$

Using these last two equations, it is seen that R, L, G, and C can be determined if γ and Z_0 are known in vector form. Z_0 can be readily found from equation 7–18 since Z_{ss} and Z_{so} are known. The quantity γl, and thus γ, may be obtained from equation 7–19 as follows: Let $\tanh \gamma l = \sqrt{Z_{ss}/Z_{so}} = M + jN$. Then $\gamma l = a + jb$ can be found by using equations A–30 and A–31. From the values of a and b, $\gamma (= \alpha + j\beta)$ is found from equations,

$$a = \alpha l$$

$$b = \beta l$$

The fact that equation A–31 is multivalued must be kept in mind, and in order to definitely fix the value of β the approximate value of the wavelength must be known.

60. Illustrative Example. An illustrative example will make the above procedure more clear. Let it be required to find the line constants of a 25-mile line on which $\omega = 5000$, ($f = 796$) and

$$Z_{ss} = 3220 \underline{/-79.29^\circ} \text{ ohms}$$

$$Z_{so} = 1301 \underline{/76.67^\circ} \text{ ohms}$$

From equation 7–18,

$$Z_0 = \sqrt{Z_{ss}Z_{so}} = \sqrt{3220 \underline{/-79.29^\circ} \times 1301 \underline{/76.67^\circ}}$$

$$= 2047 \underline{/-1.31^\circ} \text{ ohms}$$

and

$$\tanh \gamma l = M + jN = \sqrt{\frac{Z_{ss}}{Z_{so}}}$$

$$= \sqrt{\frac{3220 \underline{/-79.29^\circ}}{1301 \underline{/76.67^\circ}}}$$

$$= 1573 \underline{/-77.98^\circ}$$

$$= 0.328 - j1.54$$

Thus $M = 0.328$, $N = -1.54$, and $M^2 + N^2 = 2.479$. From equation A–30,

$$\tanh 2a = \frac{2 \times 0.328}{1 + 2.479} = 0.1886$$

$$2a = 0.1909$$

$$a = 0.0955$$

and

$$\alpha = \frac{0.0955}{25} = 0.00382 \text{ neper/mile}$$

From equation A–31,

$$\tan 2b = \frac{-2 \times 1.54}{1 - 2.479} = \frac{-3.08}{-1.479} = 2.082$$

Hence, since both numerator and denominator of the tangent function are negative, the angle $2b$ lies in the third, seventh, etc., quadrants and

$$2b = 244.35^\circ \quad \text{or} \quad (244.35^\circ + n2\pi)$$

where $n = 0, 1, 2, 3$, etc. Since this is an open-wire loaded line, the velocity of propagation should be of the order of 50,000 miles per second. The frequency of the transmitted wave being 796 cycles per second, the wavelength is about 60 or 70 miles. A 25-mile section should have a value of b

in the neighborhood of $\frac{25}{65}(360°) \doteq 140°$ or about one half of the 244.35° found above. Using this figure to determine b

$$b = 122.2° = 2.135 \text{ radians}$$

and

$$\beta = \frac{2.135}{25} = 0.0854 \text{ radian/mile}$$

Having found α and β

$$\gamma = 0.00382 + j0.0854 = 0.0855\underline{/87.43°}$$

Using Z_0, which was found above, and equation 7–20

$$\gamma Z_0 = R + j\omega L = 0.0855\underline{/87.43°} \times 2047\underline{/-1.31°}$$

$$= 175\underline{/86.12°}$$

$$= 11.83 + j175$$

from which, $R = 11.83$ ohms/mile and $L = 175/5000 = 0.0350$ henry/mile. Using equation 7–21

$$\frac{\gamma}{Z_0} = G + j\omega C = \frac{0.0855\underline{/87.43°}}{2047\underline{/-1.31°}}$$

$$= 41.8 \times 10^{-6}\underline{/88.74°}$$

$$= (0.919 + j41.8) \times 10^{-6}$$

Thus $G = 0.919 \times 10^{-6}$ mho/mile and $C = 0.00835$ μf/mile.

61. Loading. The loading of a line was previously handled (see Art. 42) by dividing the inserted inductance into per-mile units and adding these units to the regular series inductance. To treat the problem more accurately the section of the line between loading coils should be reduced to its equivalent T section, whereupon one-half the series impedance of each loading coil is added to each series arm of the section. This new T section will then allow the recalculation of new values of γ, Z_0, etc. Again use the notation of Art. 58.

$$Z_2 = \frac{Z_0}{\sinh \gamma l} \qquad [7\text{–}13]$$

$$\frac{Z_1}{2} = Z_0 \frac{\cosh \gamma l - 1}{\sinh \gamma l} \qquad [7\text{–}14]$$

From these two equations the following may be obtained:

$$\frac{Z_1}{2Z_2} = \cosh \gamma l - 1$$

or

$$\cosh \gamma l = 1 + \frac{Z_1}{2Z_2} \tag{7-22}$$

[Note that equation 7–14 can be taken as representing the series arm of the section for the unloaded case.] Let $Z_1'/2$ represent the series arm for the loaded condition and $Z_L/2$ represent the impedance to be added to each arm, that is, one-half the added loading per section. Then for the loaded section

$$\frac{Z_1'}{2} = \frac{Z_L}{2} + \frac{Z_1}{2} = \frac{Z_L}{2} + Z_0 \frac{\cosh \gamma l - 1}{\sinh \gamma l} \tag{7-23}$$

Using γ' to represent the propagation constant for the loaded condition, equation 7–22 becomes

$$\cosh \gamma' l = 1 + \frac{Z_1'}{2Z_2}$$

Substituting from equations 7–23 and 7–13

$$\cosh \gamma' l = 1 + \frac{\dfrac{Z_L}{2} + Z_0 \dfrac{\cosh \gamma l - 1}{\sinh \gamma l}}{\dfrac{Z_0}{\sinh \gamma l}}$$

$$= 1 + \frac{Z_L}{2Z_0} \sinh \gamma l + \cosh \gamma l - 1$$

$$= \cosh \gamma l + \frac{Z_L}{2Z_0} \sinh \gamma l \tag{7-24}$$

Thus the loading gives rise to a correction term,

$$\frac{Z_L}{2Z_0} \sinh \gamma l$$

Equation 7–24 is known as Campbell's Equation.

PROBLEMS

Except for Prob. 7–12, make all calculations at 796 cycles per second.

7–1. A 100-mile length of line A of Prob. 5–1 is connected between a generator whose internal impedance is $Z_g = 100 + j300$ ohms and a load impedance of $Z_r =$ 600 ohms resistance. Find the reflection factors and the insertion loss.

7–2. Determine the elements of the T section equivalent to 200 miles of line A of Prob. 5–1. Use the exact method.

7-3. Determine the elements of the π section equivalent to 200 miles of line A of Prob. 5-1. Use the exact method.

7-4. A 50-mile length of line C of Prob. 5-5 is connected between a generator whose internal impedance is $Z_g = 100 + j100$ ohms and a load impedance of $Z_r =$ 500 ohms resistance. Find the reflection factors and the insertion loss.

7-5. Find the elements of a π section equivalent to the line of Prob. 7-4.

7-6. Find the line parameters for a 50-mile line whose open- and short-circuit impedances at 796 cycles per second are

$$Z_{so} = 200\underline{/-38.9^\circ} \text{ ohms} \qquad Z_{ss} = 2440\underline{/19.81^\circ} \text{ ohms}$$

7-7. Calculate the values of Z_0, α, β, v, and λ for the line of Prob. 7-6.

7-8. Find the parameters for a 35-mile line whose open- and short-circuit impedances at 796 cycles per second are

$$Z_{so} = 268\underline{/-75.51^\circ} \text{ ohms} \qquad Z_{ss} = 604\underline{/49.91^\circ} \text{ ohms}$$

7-9. Calculate the values of Z_0, α, β, v, and λ for the line of Prob. 7-8.

7-10. Assume that the line C of Prob. 5-5 has the following loading added at intervals of 1.1 miles: $L = 175$ mh, $R = 14.3$ ohms. Calculate the new value of γ.

7-11. Assume that the line C of Prob. 5-5 has the following loading added at intervals of 1.135 miles: $L = 43$ mh, $R = 4.1$ ohms. Calculate the new value of γ.

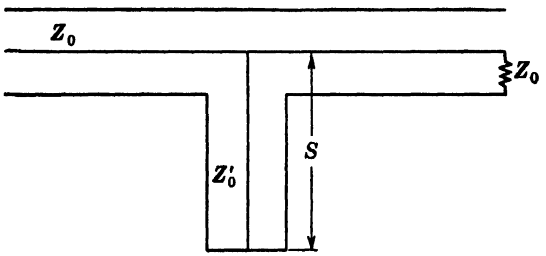


FIG. 7-4. For use in connection with Prob 7-12.

7-12. A coaxial cable is constructed as shown in Fig. 7-4. $Z_0 = 62\underline{/0^\circ}$ ohms, $Z_0' = 75\underline{/0^\circ}$ ohms, S = one-quarter wavelength at 10^9 cycles per second. Find the insertion loss in decibels of the stub at a frequency of $f = 1.2 \times 10^9$ cycles/sec. Note: the original condition is without the short-circuited stub. In this case of course there is no loss. The stub is then added which amounts to shunting a certain impedance across the line. The insertion loss is the loss caused by adding the stub. In this problem the attenuation can be considered to be zero.

CHAPTER VIII

THE POWER-TRANSMISSION LINE — EFFICIENCIES

In the preceding chapters the general theory of telephone transmission has been outlined. At this point a question may well arise as to how power-transmission-line theory differs from that covered by this development. As has already been mentioned in the Introduction, the major points of difference are:

	Power Line	Communication Line
Efficiency	high	low
Transmitted commodity	power	information

It should be expected, then, that calculations on a typical power line, using the theory developed for communication lines, would lead to results which differ appreciably from those obtained for a typical communication line. However, it is imperative that one realize that the exact solution of the transmission line as given in Chapter V is equally applicable to the telephone and the power line. The characteristic impedance Z_0 and propagation constant γ, as well as other line constants, and efficiency will be calculated for this typical power line and these values compared with those obtained on the open-wire non-loaded communication line of Art. 35. Also, for purposes of further comparison, the efficiency of a high-frequency feeder line will be calculated. The comparisons in this chapter should be considered as qualitative only. They have been interpolated at this point merely to bring out more clearly some general aspects of different kinds of lines.

62. Line Constants — Z_0, γ, v, and λ. One of the distinctions between power- and communication-line calculations is that the polyphase power-line loop usually consists of a single line conductor and a neutral return. Since the latter carries no current under balanced conditions, the loop calculations of L, C, R, and G are based on per-wire values. The series resistance of the loop, for example, is the resistance of a single wire of the three-phase power line, and the shunt conductance of the loop is the conductance of one wire to neutral.

In calculating the loop inductance of phase A in Fig. 8–1, it is simply necessary to determine the flux linkage with wire A due to I_A. Since $I_B + I_C = -I_A$ in a balanced three-phase system, one may consider

wire B and wire C to be the return path for I_A. The per-wire inductance of wire A is then

$$L_A = \left(\frac{1}{2} + 2 \ln \frac{d}{r}\right) \times 10^{-7} \text{ henry/meter} \qquad [8\text{–}1]$$

In other words the per-wire inductance of a three-phase line is simply one-half the inductance of a pair of parallel wires.

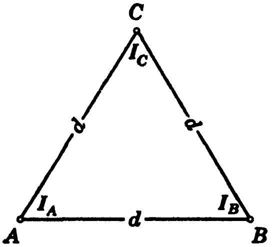


Fig. 8–1. Equilateral triangular spacing of three-phase line conductors.

The loop capacitance of one phase of a three-phase line to neutral, such as that shown in Fig. 8–1, turns out to be twice the capacitance of a pair of parallel wires of r radius separated center to center by a distance d. This value of capacitance is obtained by considering lines B and C to be one plate of the capacitor of which line A is the other plate, the justification being that for balanced three-phase voltages $Q_B + Q_C$ is at all times equal to $-Q_A$. The capacitance of line A to neutral is simply

$$C_{AN} = 2\left[\frac{1}{36 \times 10^9 \ln \frac{d}{r}}\right] \text{farad/meter} \qquad [8\text{–}2]$$

It is not the purpose of this discussion to go into the details of calculating the per phase values of L and C in polyphase lines but rather simply to outline the distinction between the per phase values of L and C in a balanced three-phase line and the values of L and C in a conventional two-wire loop. Also, if the wire configuration is other than equilateral triangular spacing as shown in Fig. 8–1, then d, the equivalent center-to-center spacing between wires, can be shown to be $\sqrt[3]{d_1 d_2 d_3}$ where d_1, d_2, and d_3 are the spacings between wires 1 and 2, 2 and 3, and 3 and 1.

In order to compare the characteristics of a power line with those of the telephone line of Art. 35, a 200-mile three-phase power line, consisting of three No. 0000 copper conductors spaced 33 inches on the corners of an equilateral triangle, will be considered. Such a line will have the following distributed parameters to neutral per mile (f = 60 cycles/sec):

$$R = 0.263 \text{ ohm (one wire)}$$

$$\omega L = 0.626 \text{ ohm (one wire)}$$

$$\omega C = 6.96\ \mu \text{ mhos}$$

$$G = 0$$

and the following characteristics:

$$z = R + j\omega L = 0.263 + j0.626 = 0.679\underline{/67.22^\circ} \text{ ohms}$$

$$y = G + j\omega C = 0 + j6.96 \times 10^{-6} = 6.96 \times 10^{-6}\underline{/90^\circ} \text{ mho}$$

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{0.679\underline{/67.22^\circ}}{6.96\underline{/90^\circ}} \times 10^6} = 100\sqrt{9.75\underline{/-22.78^\circ}}$$

$$= 312\underline{/-11.39^\circ} \text{ ohms}$$

$$\gamma = \sqrt{zy} = \sqrt{0.679\underline{/67.22^\circ} \times 6.96\underline{/90^\circ} \times 10^{-6}}$$

$$= 0.00217\underline{/78.61^\circ}$$

$$= 0.000428 + j0.00213$$

or

$$\alpha = 0.000428 \text{ neper/mile}$$

$$\beta = 0.00213 \text{ radian/mile}$$

The velocity of propagation is

$$v = \frac{\omega}{\beta} = \frac{377}{0.00213} = 177{,}000 \text{ miles/sec}$$

and the wavelength is

$$\lambda = \frac{2\pi}{\beta} = \frac{6.28}{0.00213} = 2950 \text{ miles}$$

Let these results be compared with those obtained on the nonloaded line of Art. 35.

	Telephone Line	Power Line
Z_0	745 ohms	312 ohms
α	0.00712 neper/mile	0.000428 neper/mile
β	0.0288 radian/mile	0.00213 radian/mile
v	173,500 miles/sec	177,000 miles/sec
λ	218 miles	2950 miles

It is readily apparent that the values of α and β for the power line are comparatively very low and that the wavelength is very long. The long wavelength gives rise to a predominant characteristic of a power line, that of being short electrically, that is, short in terms of wavelength. This 200-mile line which is physically long for a power line, is only 0.068 wavelength long. This means that the voltage (and current) vector in progressing the length of the line completes only a very small part of a

revolution, whereas in a long telephone line there may be many revolutions.

63. Efficiency. The low value of α for the power line is reflected in a high value of transmission efficiency. In order to compare typical power- and telephone-line efficiencies it will be necessary to calculate the efficiency of the nonloaded line of Art. 35. The received power and sending current for that line were calculated in Art. 42 and found to be 94.0 microwatts and 0.0015 ampere, respectively. Recalling that the input impedance is Z_0, the power input is given by

$$P_s = I_s^2 \times 725 = (0.0015)^2 \times 725 = 1631 \times 10^{-6} \text{ watt}$$

$$= 1631 \text{ microwatts}$$

The resistance component of Z_0 is 725 ohms.

Thus the efficiency of the telephone line becomes

$$\text{Eff} = \frac{94.0}{1631} \times 100 = 5.76\%$$

The line-to-neutral receiving-end voltage for the 200-mile power line is taken as 20,000 volts, and the three-phase load is assumed to be 1800 kilowatts at 0.8 lagging power factor. The load or receiver current is then

$$\frac{1{,}800{,}000}{3 \times 20{,}000 \times 0.8} = 37.5 \text{ amperes}$$

Hence

$$I_r = 37.5 \,/\!\underline{-36.87^\circ} \text{ amperes}$$

$$V_r = 20{,}000 \,/\!\underline{0^\circ} \text{ volts}$$

Equations 6–3 and 6–4 will be used to calculate V_s and I_s. The following are needed:

$$\sqrt{zy} = 0.00217 \,/\!\underline{78.61^\circ} \qquad \sinh a = 0.086$$

$$Z_0 = 312 \,/\!\underline{-11.39^\circ} \text{ ohms} \qquad \cosh a = 1.004$$

$$\sqrt{zy}\, S = a + jb = 0.434 \,/\!\underline{78.61^\circ} \qquad \sin b = \sin 24.5^\circ = 0.415$$

$$= 0.086 + j0.426 \qquad \cos b = 0.911$$

Equations A–24 and A–25 are used to evaluate $\sinh \sqrt{zy}\, S$ and

$\cosh \sqrt{zy}\, S.$

$$\sinh (a + jb) = \sqrt{0.0074 + 0.172} \,\Big/ \tan^{-1} \frac{0.415 \times 1.004}{0.911 \times 0.086}$$

$$= \sqrt{0.1794} \,/\tan^{-1} 5.3 = 0.424 \,/79.32°$$

$$\cosh (a + jb) = \sqrt{0.0074 + 0.83} \,\Big/ \tan^{-1} \frac{0.415 \times 0.086}{0.911 \times 1.004}$$

$$= \sqrt{0.8374} \,/\tan^{-1} 0.0391 = 0.915 \,/2.24°$$

$$V_s = 20{,}000 \times 0.915 \,/2.24° + 37.5 \,/-36.87°$$

$$\times 312 \,/-11.39° \times 0.424 \,/79.32°$$

$$= 18{,}300 \,/2.24° + 4960 \,/31.06°$$

$$= 18{,}280 + j714 + 4250 + j2560$$

$$= 22{,}530 + j3274 = 22{,}780 \,/8.27° \text{ volts}$$

$$I_s = 37.5 \,/-36.87° \times 0.915 \,/2.24° + \frac{20{,}000 \times 0.424 \,/79.32°}{312 \,/-11.39°}$$

$$= 34.3 \,/-34.63° + 27.2 \,/90.71°$$

$$= 28.2 - j19.5 - 0.332 + j27.2$$

$$= 27.9 + j7.7 = 28.9 \,/15.45° \text{ amperes}$$

The angle between the current and the voltage is 7.18°; thus the power input per phase is

$$P_s = 22{,}780 \times 28.9 \times \cos 7.18° \times 10^{-3}$$

$$= 653 \text{ kw}$$

The power output per phase is

$$P_r = \frac{1800}{3} = 600 \text{ kw}$$

Hence the efficiency is

$$\text{Eff} = \frac{600}{653} \times 100 = 92\%$$

64. Efficiency of High-Frequency Line. On a high-frequency line $\omega L \gg R$ and $\omega C \gg G$. Under these conditions Z_0 will be approximately

equal to $\sqrt{L/C}$, and, since $\sqrt{L/C}$ is dimensionally equivalent to resistance, Z_0 will be equivalent to a pure resistance R_0. Also, when the high-frequency line is used as a feeder, it will be relatively very short, and αl will then be small.

Equations 6–3 and 6–4 will be applied to a high-frequency line which is terminated in a pure resistance R_r. Since $V_r = I_r R_r$, the appropriate substitutions will be made for V_r and I_r.

$$V_s = V_r\left(\cosh \gamma l + \frac{R_0}{R_r}\sinh \gamma l\right)$$

$$= V_r\left[\left(\cosh \alpha l + \frac{R_0}{R_r}\sinh \alpha l\right)\cos \beta l + j\left(\sinh \alpha l + \frac{R_0}{R_r}\cosh \alpha l\right)\sin \beta l\right] \quad [8\text{–}3]$$

and

$$I_s = I_r\left(\cosh \gamma l + \frac{R_r}{R_0}\sinh \gamma l\right)$$

$$= I_r\left[\left(\cosh \alpha l + \frac{R_r}{R_0}\sinh \alpha l\right)\cos \beta l + j\left(\sinh \alpha l + \frac{R_r}{R_0}\cosh \alpha l\right)\sin \beta l\right] \quad [8\text{–}4]$$

The hyperbolic functions are expanded by means of equations A–20 and A–22.

The power input to the line is given by the real part of the product of V_s by the conjugate of I_s, thus:[1]

$$V_s \underset{\wedge}{I_s} = (V_s' + jV_s'')(I_s' - jI_s'')$$

$$= (V_s'I_s' + V_s''I_s'') + j(V_s''I_s' - V_s'I_s'')$$

and

$$P_s = V_s'I_s' + V_s''I_s''$$

[1] Let $V_s = V \angle \alpha$, and $I_s = I \angle \phi$. The power is then

$$VI \cos(\alpha - \phi) \quad [A]$$

Write $V_s = V(\cos\alpha + j\sin\alpha)$, and the conjugate of I_s as

$$\underset{\wedge}{I_s} = I(\cos\phi - j\sin\phi)$$

$$V_s\underset{\wedge}{I_s} = VI[(\cos\alpha\cos\phi + \sin\alpha\sin\phi) + j(\sin\alpha\cos\phi - \cos\alpha\sin\phi)]$$

The real part is

$$VI(\cos\alpha\cos\phi + \sin\alpha\sin\phi) = VI\cos(\alpha - \phi)$$

which is the same as equation A above. Thus "Power is given by the real part of the product of V_s by the conjugate of I_s."

Let m represent the ratio, R_r/R_0. Then

$$P_s = V_r I_r \left[\cosh^2 \alpha l + \sinh^2 \alpha l + \left(m + \frac{1}{m}\right) \cosh \alpha l \sinh \alpha l \right] (\cos^2 \beta l + \sin^2 \beta l)$$

$$= V_r I_r \left[\cosh 2\alpha l + \left(m + \frac{1}{m}\right) \frac{\sinh 2\alpha l}{2} \right] \qquad [8\text{–}5]$$

using equation A–19 and the identity, $\cosh a \sinh a = (\sinh 2a)/2$. Since the line is terminated in a resistance R_r, the power factor is unity and the power received is $V_r I_r$. Accordingly, the efficiency is

$$\text{Eff} = \frac{1}{\cosh 2\alpha l + \left(m + \frac{1}{m}\right) \frac{\sinh 2\alpha l}{2}} \qquad [8\text{–}6]$$

If αl is very small, $\cosh 2\alpha l \doteq 1$, and $\sinh 2\alpha l \doteq 2\alpha l$. Under this condition the efficiency is

$$\text{Eff} = \frac{1}{1 + \left(m + \frac{1}{m}\right) \alpha l}$$

Making use of the approximation given by equation 5–46, namely, $\alpha = R/(2Z_0)$

$$\text{Eff} = \frac{1}{1 + \left(m + \frac{1}{m}\right) \frac{Rl}{2R_0}} = \frac{1}{1 + \left(m + \frac{1}{m}\right) \frac{R'}{2R_0}}$$

where R' is the total resistance of the line. This may be written

$$\text{Eff} = \frac{1}{1 + \left(\frac{m^2 + 1}{m}\right) \frac{R'}{2R_0}} \qquad [8\text{–}7]$$

In order to determine the condition under which the efficiency is a maximum it is necessary to minimize the second term of the denominator with respect to m.

$$\frac{\partial \left(\frac{m^2 + 1}{m}\right)}{\partial m} = 1 - \frac{1}{m^2}$$

For maximum efficiency $\left(1 - \frac{1}{m^2}\right) = 0$ or $m = 1$. Thus maximum

efficiency occurs when $R_r = R_0$, that is, when the line is terminated in its characteristic impedance, in this case R_0. The maximum efficiency then becomes

$$\text{Max eff} = \frac{1}{1 + \dfrac{R'}{R_0}} \tag{8-8}$$

which approaches 100 per cent as R' decreases relative to R_0.

65. Illustrative Example. Consider a 1000-foot 5-megacycle feeder constructed of two parallel wires each 0.2 inch in diameter and separated 6 inches, center to center. For such a line

$$Z_0 = R_0 \doteq \sqrt{\frac{L}{C}} = \sqrt{\frac{4 \ln \dfrac{d}{r} \times 10^{-7}}{\dfrac{1}{36 \times 10^9 \ln \dfrac{d}{r}}}}$$

$$= 120 \ln \frac{d}{r} = 276 \log \frac{d}{r} \text{ ohms}$$

where d/r is the ratio of conductor separation to conductor radius. Also

$$R = \frac{135\sqrt{f} \times 10^{-4}}{r_{cm}} \text{ ohm/mile} \tag{1-82}$$

Hence

$$R'' = \frac{\sqrt{F}}{r'_{\text{in}}} \text{ ohm/1000 ft of line}$$

where F is the frequency of the transmitted wave in megacycles per second, and r'_{in} is the radius of the conductor in inches.

$$d = 6 \text{ in.}$$
$$r'_{\text{in}} = 0.1 \text{ in.}$$
$$F = 5 \text{ megacycles/sec}$$

Substituting these values into equation 8-8, the maximum efficiency per 1000 feet of line is found to be

$$\text{Eff} = \frac{1}{1 + \dfrac{\sqrt{F}/r'_{\text{in}}}{276 \log \dfrac{d}{r}}}$$

$$= \frac{1}{1 + \dfrac{\sqrt{5}/0.1}{276 \log 60}} \times 100 = 95.6\%$$

66. Summary. The efficiency of the telephone line is expected to be low because of the relatively high value of α and the length of the line. This low efficiency is partly due to the fact that the line is terminated in its characteristic impedance.

The power line on the other hand is operated in such a manner as to obtain a relatively high efficiency. No attempt is made to deliver maximum possible power. The difference in these two lines may be seen also from the fact that in the communication line the total power involved is small, and it is relatively unimportant economically whether a large proportion of it is lost or not. On the other hand the amount of power transmitted over a power line is so great that a slight difference in efficiency makes a considerable difference in the economics of the system. For instance, even in the relatively small amount of power transmitted by the above three-phase line a decrease in efficiency of 1 per cent means a loss of about 18 kilowatts. Over a period of a year, with power at ½ cent per kilowatt-hour, this amounts to a cost of approximately $800.

In the high-frequency line the assumption was made that the total αl was small. That is, αl ($= R'/2R_0$) should be low enough that the approximation,

$$\sinh 2\alpha l = \frac{R'}{R_0}$$

is justified. The hyperbolic angle can be about 0.23 before the error is 1 per cent. In the above example R'/R_0 was 0.0455. The 5-megacycle feeder could, then, be 5000 feet long, and the error would be less than 1 per cent. For the optimum efficiency condition, the line terminated in its characteristic impedance, the efficiency equation can be written

$$\text{Eff} = \frac{1}{1 + \dfrac{n\sqrt{F}}{276 r'_{\text{in}} \log \dfrac{d}{r}}} \qquad [8\text{–}9]$$

where n is the number of 1000-foot sections in the line. For a line with a given conductor separation and conductor radius this equation indicates very clearly the effect of line length and frequency of transmitted wave on the efficiency.

PROBLEMS

8–1. Determine R in ohms per mile of single conductor, L in millihenrys per mile for each conductor, and C to neutral in microfarads per mile of each conductor for No. 000 hard-drawn solid copper conductors spaced 36 inches center to center on

the corners of an equilateral triangle at a temperature of 25°C when delivering power at 60 cycles per second.

8–2. A three-phase 60-cycle-per-second power line has the following parameters, to neutral, per mile:

$$R = 0.432 \text{ ohm} \quad \text{(one wire)}$$

$$\omega L = 0.749 \text{ ohm} \quad \text{(one wire)}$$

$$\omega C = 5.76\ \mu \text{ mhos}$$

$$G = 0$$

The line is 55 miles long. Calculate Z_0, α, β, v, and λ.

8–3. Calculate the efficiency of the line of Prob. 8–2 when delivering a load of 2250 kilovolt-amperes per phase at 0.80 power factor (I lags) and 25,400 volts to neutral.

8–4. A three-phase 60-cycle-per-second power line has the following parameters, to neutral, per mile:

$$R = 0.529 \text{ ohm} \quad \text{(one wire)}$$

$$\omega L = 0.826 \text{ ohm} \quad \text{(one wire)}$$

$$\omega C = 5.19\ \mu \text{ mhos}$$

$$G = 0.244\ \mu \text{ mho}$$

Calculate Z_0, α, β, v, and λ.

8–5. Calculate the efficiency of the line of Prob. 8–4 when delivering a load of 5400 kilowatts at 58,900 volts to neutral. The power factor is 0.85 (I lags), and the length of the line is 130 miles.

8–6. A three-phase 25-cycle-per-second power line has the following parameters, to neutral, per mile:

$$R = 0.118 \text{ ohm} \quad \text{(one wire)}$$

$$\omega L = 0.334 \text{ ohm} \quad \text{(one wire)}$$

$$\omega C = 2.16\ \mu \text{ mhos}$$

$$G = 0$$

Calculate Z_0, α, β, v, and λ.

8–7. Calculate the efficiency of the line of Prob. 8–6 when delivering a load of 36,000 kilovolt-amperes per phase at 127,000 volts to neutral. The power factor is 0.90 (I lags), and the length of the line is 230 miles.

8–8. A parallel-wire line is composed of two No. 10 AWG copper conductors separated center to center by a distance of 5 inches and operates at a frequency of 1000 kilocycles per second. The distributed parameters are:

$$R = 0.0652 \text{ ohm/loop meter}$$

$$\omega L = 11.56 \text{ ohms/loop meter}$$

$$\omega C = 38.2 \times 10^{-6} \text{ mho/loop meter}$$

$$G = 0$$

The length of the line is 1000 meters and is terminated in a resistance of 1100 ohms which dissipates 227.3 watts.

(a) Find the phase angle between V_s and I_s.

(b) Find the power input to the line.

(c) Find the efficiency of the line.

8-9. A copper coaxial cable has an inner conductor of 0.1 centimeter radius, and the inner radius of the outer conductor is 0.322 centimeter. Find the efficiency of a 5-mile length of line when operating at 64 megacycles per second and terminated in a resistance equal to the characteristic impedance of the cable.

8-10. Design a 2000-foot feeder line for 3 megacycles per second using No. 2 AWG hard-drawn copper wire with the requirement that the efficiency be 95 per cent

CHAPTER IX

CONSTANT-K FILTERS

It has been shown that transmission lines in general have attenuation which is a function of frequency. A line constructed of a series of T or π sections may be made to have a definite cut-off frequency which will mark a transition between regions of good transmission and attenuation. When such a lumped-section line is constructed for the purpose of passing certain frequencies and stopping others, it is called a filter. A filter should have

(1) A given Z_0 in order to fit into a given line or between given pieces of equipment;
(2) A very low attenuation in the so-called " pass-band " and a sufficiently high attenuation in the stop or attenuation band;
(3) Given " cut-off " frequencies, frequencies which mark the dividing lines between stop and pass bands.

The applications of filters are very wide. They are commonly used in radio to eliminate unwanted frequencies, in telephone carrier systems to separate the various channels, and in power supplies to smooth out the direct current. In the last case filters are made to discriminate as much as possible against all frequencies, passing only the direct current.

In order to develop the theory of filters a beginning will be made with a consideration of some fundamental properties of reactive networks. The goal is to arrive at suitable design equations to be used when it is desired to construct a filter having certain of the requirements listed above.

67. Constant-K-Type Filters. The action of filters is based on the fact that an inductance represents a low impedance to low frequencies and a high impedance to high frequencies, whereas the opposite conditions occur with a capacitance. When inductances are connected in series on a line and capacitances in shunt, as shown in Fig. 9–1, then direct current will flow without opposition and alternating currents of low frequencies are subject to only a small impedance. As the frequency increases, the series network impedance increases, whereas that of the shunt circuit decreases. If the magnitudes of the inductances and capacitances are correct, then frequencies above a critical frequency f_0 will be attenuated, and the network forms a low-pass filter.

By a correct choice of series capacitances and shunt inductances, as shown in Fig. 9–2, a high-pass filter is formed. Such a filter cannot pass direct current and with proper design will pass only alternating currents of frequencies above a critical value f_0 without attenuation.

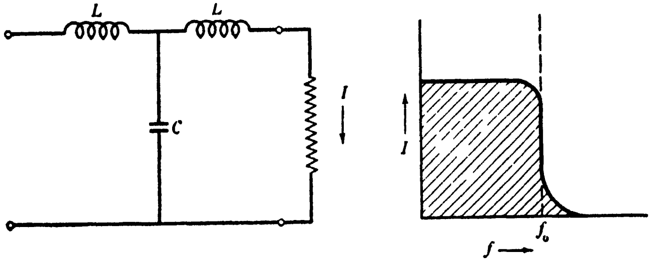


FIG 9–1. Low-pass filter

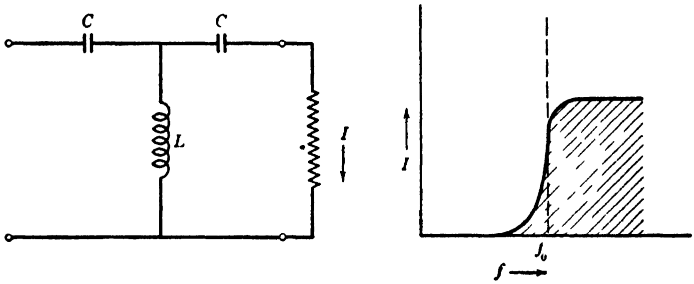


FIG 9–2. High-pass filter.

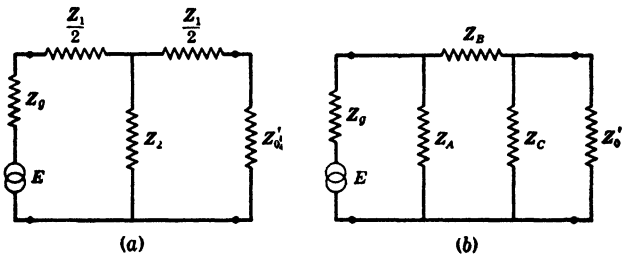


FIG 9–3 T- and π-section filters.

The networks shown in Fig. 9–3 represent simple T and π section filters. That of (a) constitutes a T section and that of (b) a π section connected between a generator and a load Z_0'. Let the load Z_0' be equal to the characteristic impedance Z_0 of the T or π section so that each network is properly terminated. The filter design problem is concerned with the propagation of power from the generator through the network and into the load Z_0'. If propagation occurs with little or no

attenuation over a certain band of frequencies then this band is called a pass band. If propagation is hindered, that is, if there occurs attenuation over a certain band, then the band is known as a stop band. When networks are constructed of pure reactances Z_1 and Z_2, such that $Z_1Z_2 = k^2$, where k is a real constant, then the network is known as a *constant-k* filter, which is one form of *prototype* filter. As an illustration, suppose $Z_1 = j\omega L_1$ and $Z_2 = -j\dfrac{1}{\omega C_2}$; then

$$Z_1Z_2 = j\omega L_1\left(-j\frac{1}{\omega C_2}\right) = \frac{L_1}{C_2} = k^2$$

It should be noted that since Z_1 and Z_2 (or Z_A, Z_B, Z_C) are pure reactances no power is absorbed by the networks themselves. Thus if the network accepts power from the generator this power is all transmitted to Z_0'. By means of elementary considerations it is evident that power will be accepted by the networks when Z_0 is a pure resistance and rejected when Z_0 is a pure reactance.

68. Stop- and Pass-Band Criteria. In order to determine the position of the stop and pass bands and to derive expressions for the frequencies at the boundaries of the bands, use will be made of equation 4–9 which gives the propagation constant γ of a properly terminated T or π section. The section is now a filter.

$$\gamma(=\alpha + j\beta) = \ln\left[1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2} + \frac{Z_1^2}{4Z_2^2}}\right] \qquad [4\text{–}9]$$

which may be written

$$\gamma = 2\ln\left[\sqrt{1 + \frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2}}\right] \qquad [9\text{–}1]$$

One of the unique features of equation 9–1 is that if

$$-1 \leqq \frac{Z_1}{4Z_2} \leqq 0 \qquad [9\text{–}2]$$

a region of zero attenuation is obtained, or, in a more precise way, the condition for zero attenuation is that

$$\frac{Z_1}{4Z_2} = A\,\underline{/\pi} \qquad [9\text{–}3]$$

where $0 \leqq A \leqq 1$. That this is true may be shown as follows on proper

substitution of equation 9–3 into equation 9–1.

$$\gamma(=\alpha+j\beta) = 2\ln\left[\sqrt{1-A}+\sqrt{-A}\right]$$
$$= 2\ln\left[\sqrt{1-A}+j\sqrt{A}\right]$$
$$= 2\ln\sqrt{1-A+A}\ \Big/ \tan^{-1}\frac{\sqrt{A}}{\sqrt{1-A}}$$
$$= 2\left[0+j\tan^{-1}\frac{\sqrt{A}}{\sqrt{1-A}}\right] \qquad [9\text{–}4]$$

Hence $\alpha = 0$ since ln 1 is zero, and

$$\beta = 2\tan^{-1}\frac{\sqrt{A}}{\sqrt{1-A}}$$

The boundary conditions are given by the extremes of $Z_1/4Z_2$, namely, 0 and -1.

$$\text{If}\quad \frac{Z_1}{4Z_2} = 0,\quad \text{then}\quad A = 0,\quad \text{and}\quad \beta = 2\tan^{-1}\frac{0}{\sqrt{1}} = 0° \qquad [9\text{–}5]$$

$$\text{If}\quad \frac{Z_1}{4Z_2} = -1,\quad \text{then}\quad A = 1,\quad \text{and}\quad \beta = 2\tan^{-1}\frac{\sqrt{1}}{0} = 180° \qquad [9\text{–}6]$$

Physically, the complex number $Z_1/4Z_2$ can have values between 0 and -1 only if Z_1 and Z_2 are reactances of opposite sign, such as for example $Z_1 = -jX_1$ and $Z_2 = jX_2$.

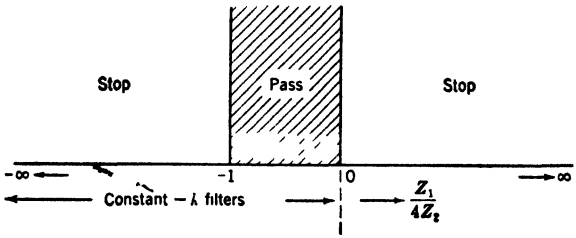


FIG. 9–4. Illustration of stop- and pass-bands.

If opposite-type reactances are employed to obtain a pass band as shown in Fig. 9–4, it is also possible for the complex number $Z_1/4Z_2$ to have values which lie between -1 and $-\infty$ but not values between 0 and $+\infty$. Hence positive values of $Z_1/4Z_2$ will not be considered in the following analysis. In the region of $Z_1/4Z_2$ which lies between -1 and $-\infty$, α takes on finite values and β remains fixed at 180°.

Let
$$\frac{Z_1}{4Z_2} = B\underline{/\pm\pi} \qquad [9\text{–}7]$$
where $B > 1$, or
$$-\infty < \frac{Z_1}{4Z_2} < -1 \qquad [9\text{–}8]$$
then
$$\begin{aligned}\alpha + j\beta &= 2 \ln [\sqrt{1 - B} + \sqrt{-B}] \\ &= 2 \ln [j\sqrt{B - 1} + j\sqrt{B}] \\ &= 2 \ln [\sqrt{B - 1} + \sqrt{B}]\underline{/\pm 90^\circ} \\ &= 2\left[\ln (\sqrt{B - 1} + \sqrt{B}) + j\left(\pm \frac{\pi}{2}\right)\right] \qquad [9\text{–}9]\end{aligned}$$

Hence $\alpha = 2 \ln (\sqrt{B - 1} + \sqrt{B})$ and is a finite value since $B > 1$, and
$$\beta = \pm\pi$$

A graphical interpretation of the results of this article is shown in Fig. 9–4. In this diagram the region to the left of the origin ($-\infty$ to 0) represents constant-k filters because in this region Z_1 and Z_2 have opposite signs, and thus when they are pure reactances their product is positive, real, and independent of frequency. In the region to the right of the origin Z_1 and Z_2 have the same sign, and thus their product will be negative, so that k would be imaginary, whereas in the constant-k filter k is a real number.

69. Illustrative Example. Given a T-section filter in which $Z_1 = j\omega L$, $Z_2 = -j/\omega C$, $L = 0.125$ henry, and $C = 0.20$ μf. Let the problem be proposed to find the frequency limits of the pass band of this filter section. These limits are given by equations 9–5 and 9–6. From equation 9–5, $Z_1/4Z_2 = 0$, it is found that $Z_1 = 0$. Therefore
$$Z_1 = j\omega L = 0$$
$$\omega = 2\pi f = 0$$
or
$$f = 0$$

For the determination of the other boundary use equation 9–6
$$\frac{Z_1}{4Z_2} = -1$$
Thus
$$j\omega L = -4Z_2 = -4 \times \frac{-j}{\omega C} = \frac{j4}{\omega C}$$

from which

$$\omega^2 = \frac{4}{LC} = \frac{4 \times 10^6}{0.125 \times 0.20}$$

$$\omega = 12{,}650 \text{ radians/sec}$$

and

$$f = \frac{\omega}{2\pi} = \frac{12{,}650}{6.28}$$

$$= 2015 \text{ cycles/sec}$$

The pass band thus extends between the frequency limits of zero and 2015 cycles per second.

70. Alternative Way of Representing Stop and Pass Bands. An alternative way of representing stop and pass bands may be found from equations 4–4, 4–5, and 7–19.

$$Z_{0T} = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} = \sqrt{Z_{ss} Z_{so}} \qquad [4\text{–}4]$$

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} \qquad [4\text{–}5]$$

where Z_{0T} and $Z_{0\pi}$ refer to the values for T and π sections respectively, and

$$\tanh \gamma = \sqrt{\frac{Z_{ss}}{Z_{so}}} \qquad [7\text{–}19]$$

where γ is the propagation constant for the entire section.

Note, in equation 4–4, that if Z_{0T} is a pure resistance then the product $Z_{ss}Z_{so}$ must be positive. This condition requires that Z_{ss} and Z_{so} have opposite signs. If Z_{0T} is a pure reactance, then Z_{ss} and Z_{so} must have the same sign. If these facts are applied to equation 7–19 the following conditions result:

(1) $\tanh \gamma$ is imaginary if Z_{0T} is a pure resistance,
(2) $\tanh \gamma$ is real if Z_{0T} is a pure reactance.

For instance, if $Z_{ss} = jX_L$ and $Z_{so} = -jX_C$, then

$$Z_{0T} = \sqrt{Z_{ss}Z_{so}} = \sqrt{X_L X_C} \quad \text{(effectively a pure resistance)}$$

and

$$\tanh \gamma = \sqrt{\frac{Z_{ss}}{Z_{so}}} = j\sqrt{\frac{X_L}{X_C}}$$

Hence $\tanh \gamma$ is imaginary. Now

$$\tanh \gamma = \frac{\sinh \alpha \cos \beta + j \cosh \alpha \sin \beta}{\cosh \alpha \cos \beta + j \sinh \alpha \sin \beta}$$

$$= \frac{\sinh \alpha \cosh \alpha + j \sin \beta \cos \beta}{\sinh^2 \alpha + \cos^2 \beta} \qquad [9\text{–}10]$$

In case 1 above, if Z_{0T} is a pure resistance the real part of equation 9–10, that is, $\dfrac{\sinh \alpha \cosh \alpha}{\sinh^2 \alpha + \cos^2 \beta}$, must be zero. This can be true only if $\alpha = 0$. Thus when Z_{0T} is a pure resistance the network offers no attenuation and a pass band occurs. On the other hand, in case 2, when Z_{0T} is a pure reactance, the imaginary part of equation 9–10 disappears, and

$$\tanh \gamma = \frac{\sinh \alpha \cosh \alpha}{\sinh^2 \alpha + \cos^2 \beta}$$

Thus, if $\tanh \gamma$ has a value, α must have a value, and there will be attenuation. (See Art. 76.)

In Figs. 9–5 and 9–6 are shown the variations of Z_{so} and Z_{ss} for low-pass and high-pass filters. From these curves which are typical of all filters, it is to be noted that the open- and short-circuit impedances change from positive to negative or vice versa as the range of frequencies is covered. Likewise the characteristic impedance changes from resistance to reactance, or in other words there are certain ranges of frequencies over which Z_0 of a given filter may be a resistance and others over which it may be a pure reactance. From the previous discussion, when the characteristic impedance is a resistance the frequencies are passed, and when the characteristic impedance is a reactance the frequencies are attenuated.

It is to be noted that if Z_1 and Z_2 in a T section are such as to make Z_{0T} a pure resistance, then $Z_{0\pi}$ is also a pure resistance because of the fact that $Z_1Z_2 = k^2$ in equation 4–5. Thus the pass bands are the same whether the elements are arranged in a T or π section.

Another way of representing pass and stop bands is to note that

$$Z_0 = \sqrt{Z_1\left(Z_2 + \frac{Z_1}{4}\right)}$$

will be a pure resistance, and this will result in a pass band, if Z_1 and $\left(Z_2 + \dfrac{Z_1}{4}\right)$ are of opposite sign, whereas if Z_1 and $\left(Z_2 + \dfrac{Z_1}{4}\right)$ are of the same sign then Z_0 will be a pure reactance and a stop band results. In

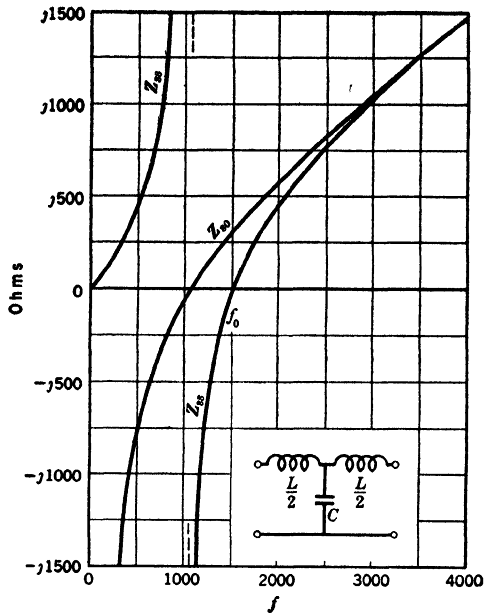


FIG. 9–5. Variation of Z_{so} and Z_{ss} with frequency for low-pass filter. $f_0 = 1500$ cycles per second, $C = 0.354\ \mu f$, $L = 0.1273$ henry.

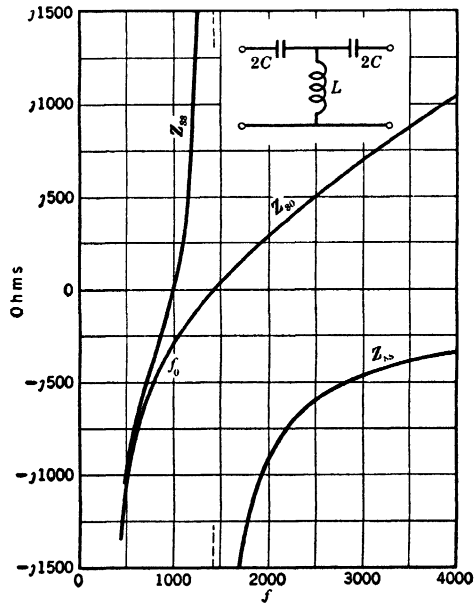


FIG. 9–6. Variation of Z_{so} and Z_{ss} with frequency for high-pass filter. $f_0 = 1000$ cycles per second, $C = 0.1326\ \mu f$, $L = 0.0477$ henry.

Fig. 9–7 are curves showing typical variations of Z_1 and $\left(Z_2 + \frac{Z_1}{4}\right)$ with frequency. In this diagram the reactances Z_1 and $\left(Z_2 + \frac{Z_1}{4}\right)$ have opposite signs between A–B and between C–D. Thus these two bands are pass bands. In the regions 0–A, B–C, and D–∞ these quantities have the same sign and over these ranges stop-band conditions exist.

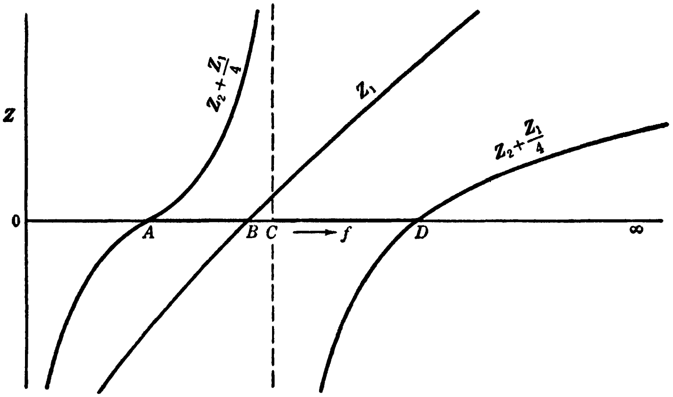


FIG. 9–7. Typical variations of Z_1 and $\left(Z_2 + \frac{Z_1}{4}\right)$ with frequency.

71. Illustrative Example. Given a T section in which $Z_1 = j\omega L$, $Z_2 = -j/\omega C$, $L = 0.125$ henry, and $C = 0.20\ \mu f$. To determine, by the method of Art. 70, whether frequencies of $f_1 = 1000$ cycles/sec and $f_2 = 2500$ cycles/sec will be passed.

First calculate Z_1 and Z_2 at the frequency f_1 of 1000 cycles per second.

$$Z_1 = j\omega L = j2\pi \times 1000 \times 0.125 = j785 \text{ ohms}$$

$$Z_2 = \frac{-j10^6}{\omega C} = \frac{-j10^6}{2\pi \times 1000 \times 0.20} = -j796 \text{ ohms}$$

From equation 4–4,

$$Z_0 = \sqrt{j785\ (-j796) + \frac{(j785)^2}{4}}$$

$$= \sqrt{624{,}900 - \frac{616{,}200}{4}}$$

$$= 686 \text{ ohms} \quad \text{(resistance)}$$

It can also be shown that for this case

$$\tanh \gamma = \sqrt{\frac{Z_{ss}}{Z_{so}}} = \frac{\omega\sqrt{LC}\sqrt{\omega^2 LC - 4}}{\omega^2 LC - 2}$$

Hence

$$\tanh \gamma = -j1.698$$

Since Z_0 is real and $\tanh \gamma$ is imaginary, α must be zero. Therefore the frequency of 1000 cycles per second will be passed.

For $f_2 = 2500$ cycles/sec

$$Z_1 = j2\pi \times 2500 \times 0.125 = j1964 \text{ ohms}$$

$$Z_2 = \frac{-j10^6}{2\pi \times 2500 \times 0.20} = -j318 \text{ ohms}$$

and

$$Z_0 = \sqrt{j1964\ (-j318) + \frac{(j1964)^2}{4}}$$

$$= \sqrt{624{,}600 - 964{,}300}$$

$$= j583 \text{ ohms} \quad \text{(reactance)}$$

Also for this case

$$\tanh \gamma = \sqrt{\frac{Z_{ss}}{Z_{so}}} = \frac{\omega\sqrt{LC}\sqrt{\omega^2 LC - 4}}{\omega^2 LC - 2}$$

$$= 0.876$$

Z_0 is now imaginary while $\tanh \gamma$ is real; therefore α has a real value, and the frequency of 2500 cycles per second will be attenuated.

72. Cut-Off Frequency. The frequencies which represent the boundary lines between pass and stop bands are called cut-off frequencies f_0 and are given by equations 9–5 and 9–6. By applying these equations to the two simple constant-k filters shown in Figs. 9–8 and 9–9 their respective cut-off frequencies are obtained. In Fig. 9–8,

$$Z_1 = j\omega L, \qquad Z_2 = \frac{-j}{\omega C}$$

and

$$\frac{Z_1}{4Z_2} = \frac{-\omega^2 LC}{4} = 0 \text{ and } -1$$

Thus

$$\omega^2 = 0 \text{ and } \frac{4}{LC}$$

and

$$\omega = 0 \text{ and } \frac{2}{\sqrt{LC}}$$

from which

$$f_0 = 0 \text{ and } \frac{1}{\pi\sqrt{LC}} \quad \text{(for a low-pass filter)} \qquad [9\text{–}11]$$

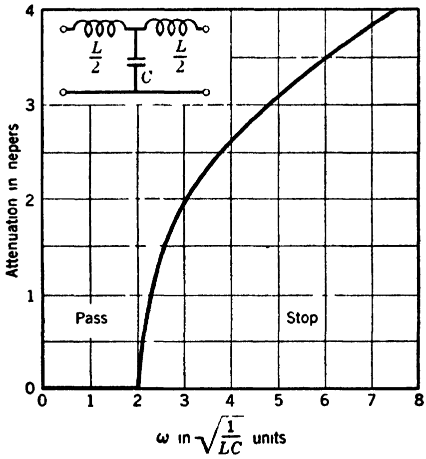


FIG 9–8. Attenuation curve for low-pass constant-k filter

In Fig. 9–9,

$$Z_1 = \frac{-j}{\omega C}, \qquad Z_2 = j\omega L$$

and

$$\frac{Z_1}{4Z_2} = \frac{-1}{4\omega^2 LC} = 0 \text{ and } -1$$

$$\omega^2 = \infty \text{ and } \frac{1}{4LC}$$

$$\omega = \infty \text{ and } \frac{1}{2\sqrt{LC}}$$

from which

$$f_0 = \infty \text{ and } \frac{1}{4\pi\sqrt{LC}} \quad \text{(for a high-pass filter)} \qquad [9\text{–}12]$$

Since the pass band lies between the cut-off frequencies, the filter shown in Fig. 9–8 is called a *low-pass* filter, as its pass band lies between

$f_0 = 0$ and $f_0 = 1/(\pi\sqrt{LC})$. The network shown in Fig. 9-9 is called a *high-pass* filter since its pass band lies between $f_0 = 1/(4\pi\sqrt{LC})$ and $f_0 = \infty$.

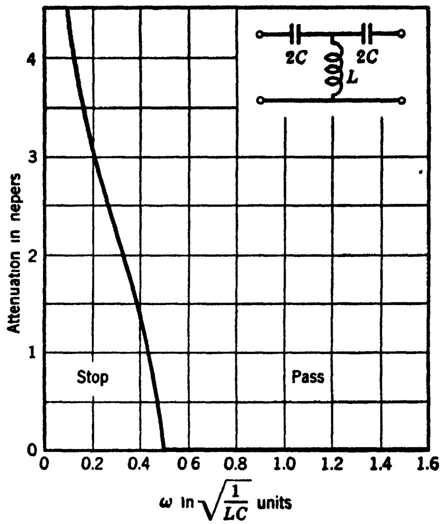


FIG. 9-9. Attenuation curve for high-pass constant-k filter.

73. Variation of Z_0 with Frequency. A knowledge of the variation of the characteristic impedance of constant-k T and π section filters with frequency is important when it becomes necessary to match these impedances.

From Art. 25,

$$Z_{0T} = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} \qquad [4\text{-}1]$$

and

$$Z_{0\pi} = \frac{Z_1Z_2}{Z_{0T}} \qquad [4\text{-}5]$$

Using again the filter sections of Figs. 9-8 and 9-9, there is obtained for the low-pass filter

$$Z_{0T} = \sqrt{\frac{-j\omega Lj}{\omega C} - \frac{\omega^2L^2}{4}}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2LC}{4}\right)}$$

Substitute $LC/4 = 1/\omega_0^2$ through use of equation 9–11, and obtain

$$Z_{0T} = \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2}{\omega_0^2}\right)} = \sqrt{\frac{L}{C}}\sqrt{1 - \frac{f^2}{f_0^2}} \quad \text{(l.p.)} \qquad [9\text{–}13]$$

Similarly

$$Z_{0\pi} = \frac{\dfrac{L}{C}}{\sqrt{\dfrac{L}{C}\left(1 - \dfrac{\omega^2}{\omega_0^2}\right)}} = \frac{\sqrt{\dfrac{L}{C}}}{\sqrt{1 - \dfrac{f^2}{f_0^2}}} \quad \text{(l.p.)} \qquad [9\text{–}14]$$

For the high-pass filter

$$Z_{0T} = \sqrt{\frac{-j\omega Lj}{\omega C} - \frac{1}{4\omega^2 C^2}}$$

$$= \sqrt{\frac{L}{C}\left(1 - \frac{1}{4\omega^2 LC}\right)}$$

Substitute $1/(LC) = 4\omega_0^2$ through use of equation 9–12 and obtain

$$Z_{0T} = \sqrt{\frac{L}{C}\left(1 - \frac{\omega_0^2}{\omega^2}\right)} = \sqrt{\frac{L}{C}}\sqrt{1 - \frac{f_0^2}{f^2}} \quad \text{(h.p.)} \qquad [9\text{–}15]$$

Similarly

$$Z_{0\pi} = \frac{\dfrac{L}{C}}{\sqrt{\dfrac{L}{C}\left(1 - \dfrac{\omega_0^2}{\omega^2}\right)}} = \frac{\sqrt{\dfrac{L}{C}}}{\sqrt{1 - \dfrac{f_0^2}{f^2}}} \quad \text{(h.p.)} \qquad [9\text{–}16]$$

The trend of these functions is shown in Fig. 9–10. It is evident from these curves that the very wide limits in the correct terminating impedance of these filters give rise to serious difficulties in communication circuits. Obviously for a given receiving impedance there is only one frequency for which the section is correctly terminated.

74. Design Equations for Constant-K Filters. It will be noted from Art. 73 that Z_0 is a function of frequency, and thus, if a filter is to be designed to match a certain line, it can be done only at one frequency unless the line impedance is also a similar function of the frequency. Hence, there may be mismatching at all frequencies but one in the pass band and thus a consequent loss. This problem will be taken up later. For purposes of the immediate design, zero frequency will be used for calculating Z_0 of the low-pass filter, while $f = \infty$ will be used for the high-pass filter. Also it is seen that over the pass band Z_0 is a pure resistance

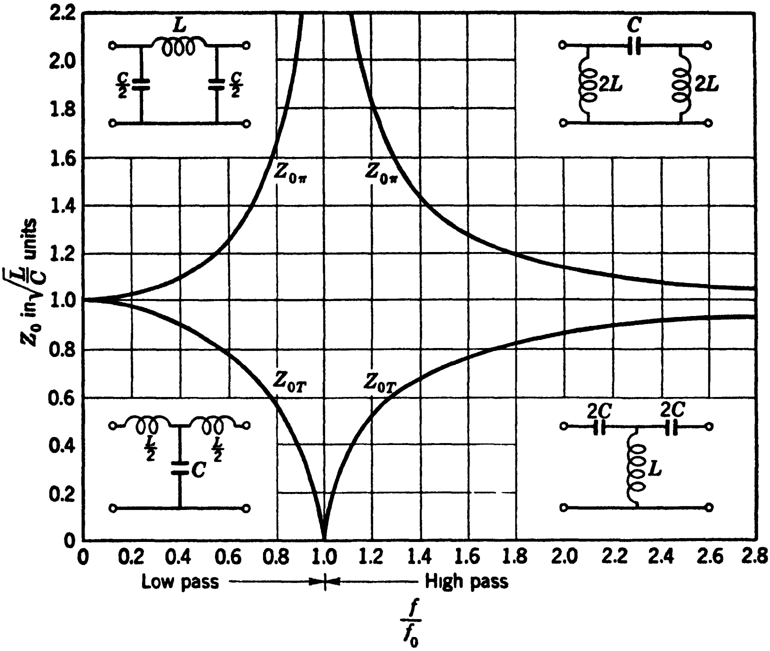


FIG. 9–10. Variation of Z_{0T} and $Z_{0\pi}$ of low- and high-pass filters with frequency over the respective pass bands.

which, for these two particular frequencies, is equal to $\sqrt{L/C}$ and as such will be indicated by R_0. As a consequence, for both the low- and high-pass filters and for T and π sections

$$R_0 = \sqrt{\frac{L}{C}}$$

For the low-pass filter

$$f_0 = \frac{1}{\pi\sqrt{LC}}$$

Solving these two equations for L and C

$$L = R_0^2 C$$

$$L = \frac{1}{\pi^2 f_0^2 C}$$

Therefore

$$R_0^2 C^2 = \frac{1}{\pi^2 f_0^2}, \qquad C^2 = \frac{1}{\pi^2 f_0^2 R_0^2}$$

and

$$C = \frac{1}{\pi f_0 R_0} \text{ farad} \quad \text{(l.p.)} \tag{9-17}$$

Also

$$L = \frac{R_0}{\pi f_0} \text{ henry} \quad \text{(l.p.)} \tag{9-18}$$

For the high-pass filter

$$R_0 = \sqrt{\frac{L}{C}}$$

$$f_0 = \frac{1}{4\pi\sqrt{LC}}$$

from which

$$L = R_0^2 C$$

$$L = \frac{1}{4^2\pi^2 f_0^2 C}$$

Therefore

$$R_0^2 C^2 = \frac{1}{4^2\pi^2 f_0^2}, \qquad C^2 = \frac{1}{4^2\pi^2 f_0^2 R_0^2}$$

and

$$C = \frac{1}{4\pi f_0 R_0} \text{ farad} \quad \text{(h.p.)} \tag{9-19}$$

$$L = \frac{R_0}{4\pi f_0} \text{ henry} \quad \text{(h.p.)} \tag{9-20}$$

75. Illustrative Example. Let it be required to design a constant-k type high-pass filter with a cut-off frequency at $f_0 = 1000$ cycles/sec and a characteristic impedance at $f = \infty$ of $Z_0 = 600$ ohms. The design equations 9–19 and 9–20 may be used directly.

$$C = \frac{10^6}{4\pi \times 1000 \times 600} = 0.1326\ \mu\text{f}$$

$$L = \frac{600}{4\pi \times 1000} = 0.0477 \text{ henry}$$

$$= 47.7 \text{ mh}$$

See Fig. 9–6.

76. Attenuation and Phase Shift of Low- and High-Pass Filters. The propagation constant for a general T or π section, as a function of Z_1

and Z_2 has been shown to be given by

$$\gamma(=\alpha + j\beta) = \ln\left[1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2} + \frac{Z_1^2}{4Z_2^2}}\right] \qquad [4\text{–}9]$$

$$= \cosh^{-1}\left(1 + \frac{Z_1}{2Z_2}\right)$$

or

$$\cosh \gamma = 1 + \frac{Z_1}{2Z_2} \qquad [7\text{–}22]$$

Since Z_1 and Z_2 are pure reactances, Z_1/Z_2 is real so that

$$\cosh \gamma = \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta = \text{(a real quantity)}$$

Since $\cosh \gamma$ is real, the imaginary term is zero, and

$$\cosh \gamma = \cosh \alpha \cos \beta \qquad [9\text{–}21]$$

Let the above equations be applied first to a low-pass constant-k filter as given in Fig. 9–8. Here $Z_1 = j\omega L$, and $Z_2 = -j/(\omega C)$. Hence

$$\cosh \gamma = \cosh \alpha \cos \beta = 1 + \frac{Z_1}{2Z_2} = 1 - \frac{\omega^2 LC}{2} \qquad [9\text{–}22]$$

It has been shown that, over the pass band, $\alpha = 0$. Thus for this range

$$\cosh \gamma = \cos \beta = 1 - \frac{\omega^2 LC}{2}$$

and

$$\beta = \cos^{-1}\left(1 - \frac{\omega^2 LC}{2}\right) \qquad [9\text{–}23]$$

where $\omega^2 LC/2$ must be less than unity. The relationship between β and ω is shown in Fig. 9–11. Note that at the low-frequency end the phase shift is 0° and that it increases to 180° by the time the edge of the pass band has been reached.

It has been shown in Art. 68 that beyond the limit of the pass band the phase shift remains at 180°. Equation 9–22 then becomes

$$\cosh \gamma = -\cosh \alpha = 1 - \frac{\omega^2 LC}{2}$$

$$\cosh \alpha = \frac{\omega^2 LC}{2} - 1$$

Thus:

$$\alpha = \cosh^{-1}\left(\frac{\omega^2 LC}{2} - 1\right) \qquad [9\text{–}24]$$

The relationship between α and ω is shown in Fig. 9–8.

For the high-pass filter of Fig. 9–9, $Z_1 = -j/(\omega C)$, and $Z_2 = j\omega L$. Then:

$$\cosh \gamma = \cosh \alpha \cos \beta = 1 - \frac{1}{2\omega^2 LC} \qquad [9\text{–}25]$$

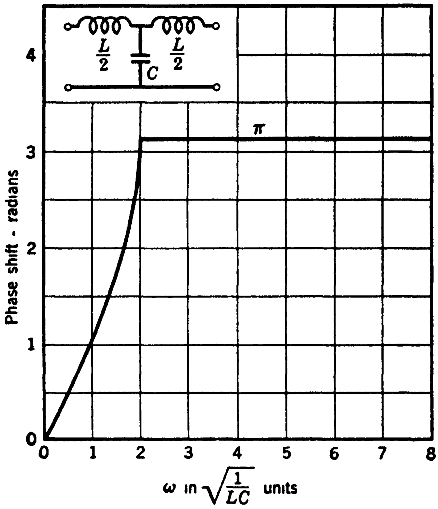


FIG. 9–11. Phase shift of low-pass constant-k filter.

because cosh γ is again real. As ω is increased from 0 to ∞, the attenuation band is gone through first and through it the value of β is 180°. Thus equation 9–25 becomes

$$\cosh \alpha = \frac{1}{2\omega^2 LC} - 1$$

or

$$\alpha = \cosh^{-1}\left(\frac{1}{2\omega^2 LC} - 1\right) \qquad [9\text{–}26]$$

Over the pass band α is zero, so equation 9–25 becomes

$$\cos \beta = 1 - \frac{1}{2\omega^2 LC}$$

or

$$\beta = \cos^{-1}\left(1 - \frac{1}{2\omega^2 LC}\right) \qquad [9\text{–}27]$$

The relationship between α and ω is shown in Fig. 9–9 and that between β and ω in Fig. 9–12.

77. The Band-Pass Filter. A band-pass filter is one whose pass band lies between two definite frequencies f_0' and f_0'' neither one of which is zero. Obviously this effect can be obtained by placing a high-pass

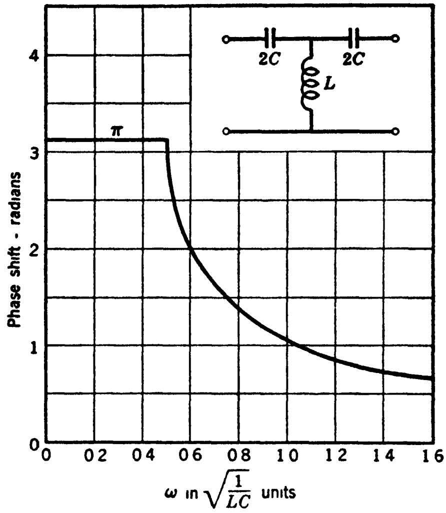


FIG. 9–12. Phase shift of high-pass constant-k filter.

filter in tandem with a low-pass filter such that the cut-off frequency of the low-pass filter is higher than that of the high-pass filter. In this case the attenuation characteristics should be as shown in Fig. 9–13.

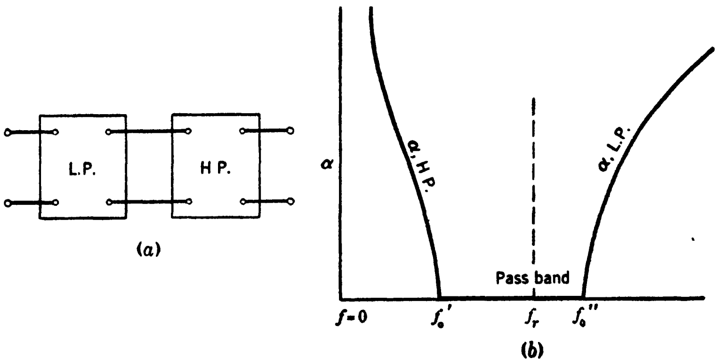


FIG. 9–13. Attenuation in band-pass filter.

The same general effect can be obtained by the use of the network of Fig. 9–14 wherein any frequency which will produce resonance in L_1, C_1 and L_2, C_2 simultaneously will be passed easily because the series impedance becomes zero and the shunt impedance infinite.

Let us investigate this circuit in some detail. In order to apply the filter theory it is necessary to write

$$Z_1 = j\omega L_1 - \frac{j}{\omega C_1}$$

$$Z_2 = \frac{\dfrac{-j\omega L_2 j}{\omega C_2}}{j\omega L_2 - \dfrac{j}{\omega C_2}}$$

and

$$Z_1 Z_2 = \frac{\dfrac{L_2}{C_2}\left(\omega L_1 - \dfrac{1}{\omega C_1}\right)}{\omega L_2 - \dfrac{1}{\omega C_2}} = \frac{L_2(\omega^2 L_1 C_1 - 1)}{C_1(\omega^2 L_2 C_2 - 1)}$$

and the filter will be a constant-k type provided

$$L_1 C_1 = L_2 C_2 \qquad [9\text{–}28]$$

since this equality makes $Z_1 Z_2$ independent of frequency.

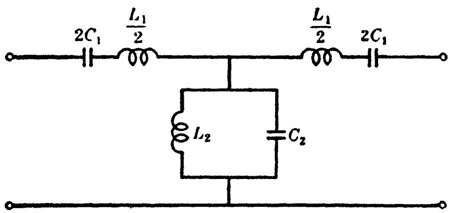


FIG. 9–14. Band-pass filter, T section.

In order to find the cut-off frequencies calculate the value of $Z_1/4Z_2$ and set equal to -1 and 0 successively.

$$\frac{Z_1}{4Z_2} = \frac{-\left(\omega_0 L_1 - \dfrac{1}{\omega_0 C_1}\right)\left(\omega_0 L_2 - \dfrac{1}{\omega_0 C_2}\right)}{4 \cdot \dfrac{L_2}{C_2}} = -1 \text{ and } 0 \qquad [9\text{–}29]$$

Using the first equality and making the substitution, $L_2 C_2 = L_1 C_1$, and $L_2 = L_1 C_1 / C_2$ results in

$$\frac{C_2(\omega_0^2 L_1 C_1 - 1)^2}{4\omega_0^2 L_1 C_1^2} = 1$$

from which

$$\omega_0 = \sqrt{\frac{1}{L_1C_1} + \frac{1}{L_1C_2} \pm \frac{1}{\sqrt{L_1C_2}}} \qquad [9\text{–}30]$$

Equation 9–30 indicates that there are lower and upper cut-off frequencies as follows:

$$f_0' = \frac{1}{2\pi}\left(\sqrt{\frac{1}{L_1C_1} + \frac{1}{L_1C_2} - \frac{1}{\sqrt{L_1C_2}}}\right)$$

and

$$f_0'' = \frac{1}{2\pi}\left(\sqrt{\frac{1}{L_1C_1} + \frac{1}{L_1C_2} + \frac{1}{\sqrt{L_1C_2}}}\right) \qquad [9\text{–}31]$$

The second possible equality from equation 9–29 gives the equation,

$$(\omega^2L_1C_1 - 1)^2 = 0$$

or

$$f_r = \frac{1}{2\pi\sqrt{L_1C_1}} \qquad [9\text{–}32]$$

This is the frequency of resonance of Z_1 and Z_2 and can be shown to be the geometric mean between f_0' and f_0'' as follows: Let f_0' and f_0'' be multiplied together, and obtain

$$f_0'f_0'' = \frac{1}{4\pi^2}\left[\frac{1}{L_1C_1} + \frac{1}{L_1C_2} - \frac{1}{L_1C_2}\right] = \frac{1}{4\pi^2L_1C_1} \qquad [9\text{–}33]$$

and the geometric mean is

$$\sqrt{f_0'f_0''} = \frac{1}{2\pi\sqrt{L_1C_1}} \qquad [9\text{–}34]$$

which is f_r of equation 9–32. According to previous theory the pass bands are two in number and lie between f_r and f_0' on one side and between f_r and f_0'' on the other. The situation is shown in Fig. 9–13*b*. Thus there are actually two pass bands which on account of the condition that $L_1C_1 = L_2C_2$ join at f_r so that there is a continuous pass band from f_0' to f_0''.

The expression for the characteristic impedance of such a filter is

$$Z_0 = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}}$$

$$= \sqrt{\frac{L_2}{C_1} + \frac{Z_1^2}{4}}$$

The value of Z_0 is taken at the frequency at which the Z_1 term drops out, that is, at f_r.

$$Z_0 = \sqrt{\frac{L_2}{C_1}} = \sqrt{\frac{L_1}{C_2}} = R_0 \qquad [9\text{–}35]$$

In order to write the design equations for a band-pass filter, first form the difference $f_0'' - f_0'$ from equation 9–31,

$$f_0'' - f_0' = \frac{1}{\pi\sqrt{L_1C_2}} \qquad [9\text{–}36]$$

From equation 9–35

$$L_1 = C_2R_0^2$$

and from equation 9–36,

$$L_1 = \frac{1}{C_2(f_0'' - f_0')^2\pi^2}$$

therefore

$$C_2 = \frac{1}{\pi R_0(f_0'' - f_0')} \qquad [9\text{–}37]$$

$$L_1 = \frac{R_0}{\pi(f_0'' - f_0')} \qquad [9\text{–}38]$$

From equation 9–34,

$$f_0'f_0'' = \frac{1}{4\pi^2L_1C_1} \qquad [9\text{–}39]$$

From equations 9–38 and 9–39,

$$C_1 = \frac{f_0'' - f_0'}{4\pi f_0'f_0''R_0} \qquad [9\text{–}40]$$

and, from equation 9–35,

$$L_2 = C_1R_0^2 = \frac{R_0(f_0'' - f_0')}{4\pi f_0'f_0''} \qquad [9\text{–}41]$$

Equations 9–37, 9–38, 9–40, and 9–41 are the desired design equations in terms of R_0 and the two cut-off frequencies.

Illustrative Example. Let it be required to design a band-pass filter with the following characteristics: $Z_0 = R_0 = 600$ ohms, $f_0' = 1000$ cycles/sec, $f_0'' = 2000$ cycles/sec. Equations 9–37, 9–38, 9–40, and 9–41 will be used.

$$C_2 \doteq \frac{10^6}{\pi 600 \times (2000 - 1000)} = \frac{10^6}{600{,}000\pi} = 0.530\ \mu f$$

$$L_1 = \frac{600}{\pi(2000 - 1000)} = 0.191 \text{ henry}$$

$$C_1 = \frac{(2000 - 1000) \times 10^6}{4\pi \times 1000 \times 2000 \times 600} = \frac{10^6}{4\pi \times 1,200,000} = 0.0663\ \mu\text{f}$$

$$L_2 = \frac{600 \times (2000 - 1000)}{4\pi \times 1000 \times 2000} = \frac{600}{4\pi \times 2000} = 0.02385\ \text{henry}$$

This filter is shown in Fig. 9-15. Its pass band lies between 1000 and 2000 cycles per second, and at the frequency of 1414 cycles per second, the geometric mean of the two cut-off frequencies, it will match an incoming and out-

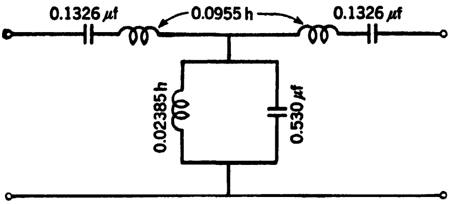


FIG. 9-15. For use in connection with illustrative example of Art. 77.

going line whose characteristic impedance is 600 ohms. It must be kept in mind that this filter will not match a 600-ohm impedance at any other frequency.

78. The Band Elimination Filter. If it is desired to stop a band of frequencies and to allow all others to pass, a possible configuration can easily be set up on the basis of the previous discussion. The result

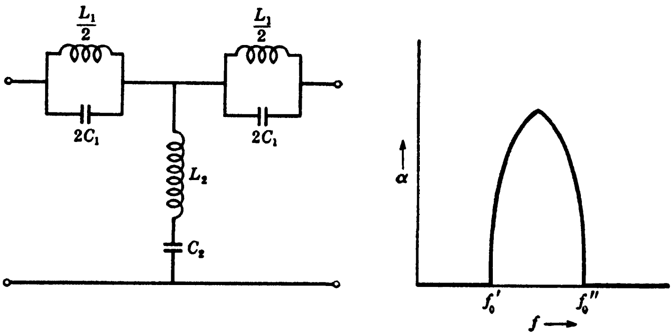


FIG. 9-16. Band-elimination filter, T section.

may be attained by placing a high-pass and a low-pass filter in parallel, overlapping their stop bands in such a way as to stop only a limited range of frequencies.

However, using the same principle as in Art. 77, a relatively simple circuit, such as the T section of Fig. 9-16, can be set up, and the possi-

bility of its performing in a satisfactory manner can be investigated. It is left as an exercise for the student to work out the details for determining:

$$L_1 = \frac{R_0(f_0'' - f_0')}{\pi f_0' f_0''}$$

$$C_1 = \frac{1}{4\pi R_0(f_0'' - f_0')}$$

$$L_2 = \frac{R_0}{4\pi(f_0'' - f_0')}$$

$$C_2 = \frac{f_0'' - f_0'}{\pi f_0' f_0'' R_0}$$

The lower and upper cut-off frequencies are

$$f_0' = \frac{1}{8\pi}\left(\sqrt{\frac{1}{L_2C_1} + \frac{16}{L_1C_1}} - \frac{1}{\sqrt{L_2C_1}}\right)$$

$$f_0'' = \frac{1}{8\pi}\left(\sqrt{\frac{1}{L_2C_1} + \frac{16}{L_1C_1}} + \frac{1}{\sqrt{L_2C_1}}\right)$$

The frequency for the calculation of Z_0 is

$$f = \sqrt{f_0' f_0''} = \frac{1}{2\pi\sqrt{L_1C_1}} = \frac{1}{2\pi\sqrt{L_2C_2}}$$

79. Effect of Resistance in Filter Elements. The theory of filters as presented in this chapter has not taken into consideration the effect of any resistance, or power dissipation, occurring within the filter elements. Actually any inductive element may be expected to have appreciable resistance although resistance which occurs in conjunction with capacitance may be assumed to be negligible. For the purposes of the present treatment it is sufficient to point out the effect of such resistance and to note the limitations produced by it.

It is clear that, in general, power dissipation in a filter will produce attenuation at all frequencies. Thus the pass band, instead of presenting zero attenuation, as was shown in the previous articles, would present a certain rather low attenuation which cannot be eliminated.

This effect becomes more pronounced at the regions of the cut-off frequencies. Also at frequencies of infinite attenuation one would expect a reduction in attenuation for much the same reason that the effectiveness of a resonant parallel L–C circuit is reduced on account of the resistance present.[1]

The general effect of resistance on the attenuation curve of a filter is shown in Fig. 9–17.

If a filter is placed in a line the loss at all frequencies, except those for which the filter matches the line, will be greater than that shown in Fig. 9–17 on account of the fact that there is a mismatch of impedances.

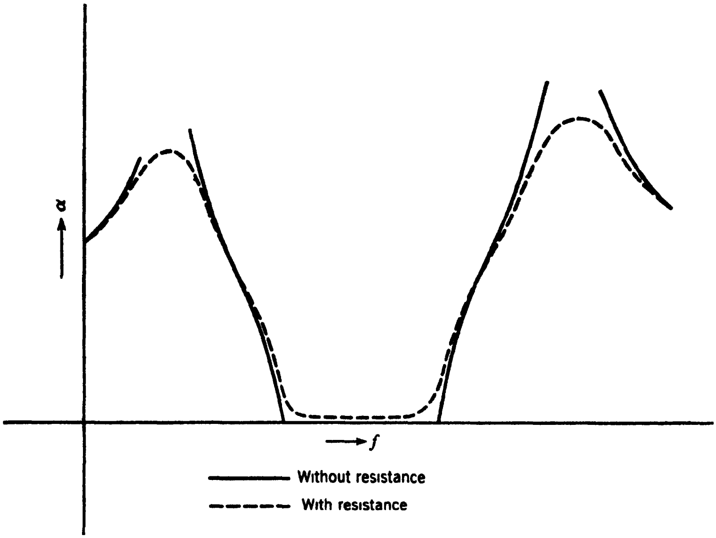


FIG. 9–17. Effect of resistance in filters.

80. Single-Frequency Rejection Filter. A special filter circuit which does not come under the heading of constant k, but which is very useful for certain purposes, such as the elimination of a single frequency, will be described. This circuit is shown in Fig. 9–18, and its operation depends on the fact that the R–C π section at a–b–c is equivalent to a T section in which the shunt arm Z_3 has a negative-resistance component which can be used to cancel the resistance r of the coil.

The π section a–b–c of Fig. 9–18 is transformed, by means of equations 2–4, 2–5, and 2–6, into a T section. The equivalent T section is shown

[1] Reed, H. R., and G. F. Corcoran, *Electrical Engineering Experiments*, pp. 235–236, New York, John Wiley & Sons, 1939.

at 1–2–3 of Fig. 9–19 where

$$Z_1 = Z_2 = \frac{2R}{R^2\omega^2C^2 + 4} - j\frac{R^2\omega C}{R^2\omega^2C^2 + 4} \tag{9–42}$$

$$Z_3 = -\frac{R}{R^2\omega^2C^2 + 4} - j\frac{\dfrac{2}{\omega C}}{R^2\omega^2C^2 + 4} \tag{9–43}$$

Thus it is seen that the shunt branch is composed of an inductance, a capacitance, and two resistances, one of which is negative. The condi-

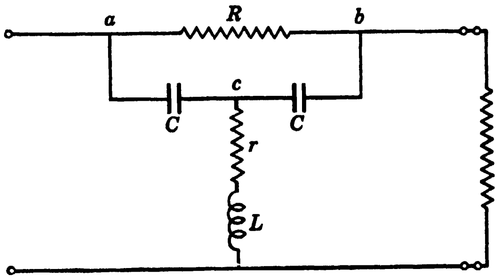


FIG. 9–18. Single-frequency rejection filter.

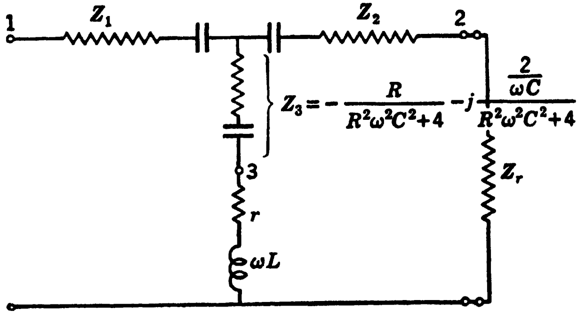


FIG. 9–19. Equivalent circuit of Fig. 9–18.

tion for infinite attenuation of the section is given by the following equations, which express resonance, without a resistance component, of the shunt branch,

$$r = \frac{R}{R^2\omega^2C^2 + 4} \tag{9–44}$$

and

$$\omega L = \frac{\dfrac{2}{\omega C}}{R^2\omega^2C^2 + 4} \tag{9–45}$$

In many practical cases it is permissible to assume that

$$R^2\omega^2C^2 \ll 4 \qquad [9\text{–}46]$$

in which case

$$r \doteq \frac{R}{4}$$

and

$$\omega L \doteq \frac{1}{2\omega C}$$

or

$$R \doteq 4r \qquad [9\text{–}47]$$

and

$$C \doteq \frac{1}{2\omega^2 L} \qquad [9\text{–}48]$$

In the vicinity of the frequency of total rejection, the filter section thus becomes that shown in Fig. 9–20. By means of equations 9–47

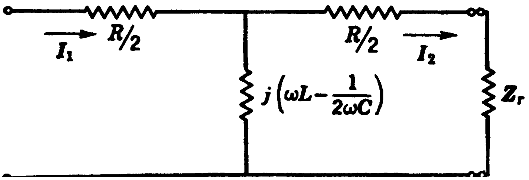


FIG. 9–20. Equivalent circuit of Fig. 9–19 when $R^2\omega^2C^2 \ll 4$ or $\omega^2L^2/r^2 \gg 1$.

and 9–48 it is readily seen that the approximation $R^2\omega^2C^2 \ll 4$ is equivalent to the condition of $\omega^2L^2/r^2 \gg 1$. Thus the necessary condition for determining whether the approximate circuit is valid is converted from the usually unknown R and C to the usually known r and L.

As an illustration, suppose that it is necessary to reject completely a frequency of 5000 cycles per second from an impedance Z_r, using a coil having an inductance of 10 millihenrys and a resistance of 30 ohms. What are the values of R and C to be used in the circuit of Fig. 9–18?

$$\omega = 2\pi f = 2 \times \pi \times 5000 = 10{,}000\pi$$

$$L = 0.010 \text{ henry}$$

$$r = 30 \text{ ohms}$$

In order to use the equivalent circuit of Fig. 9–20, is $\omega^2L^2/r^2 \gg 1$?

$$\omega^2L^2/r^2 = 109.6 \quad \text{(which is substantially larger than 1)}$$

From equation 9–47,

$$R = 4 \times 30 = 120 \text{ ohms}$$

From equation 9–48,

$$C = \frac{1}{2(10{,}000\pi)^2 \times 0.01} = 0.0507 \times 10^{-6} \quad \text{farad}$$

Since under the above conditions, the shunt branch will be a short circuit to currents having a frequency of 5000 cycles per second, there will be total rejection at 5000 cycles per second.

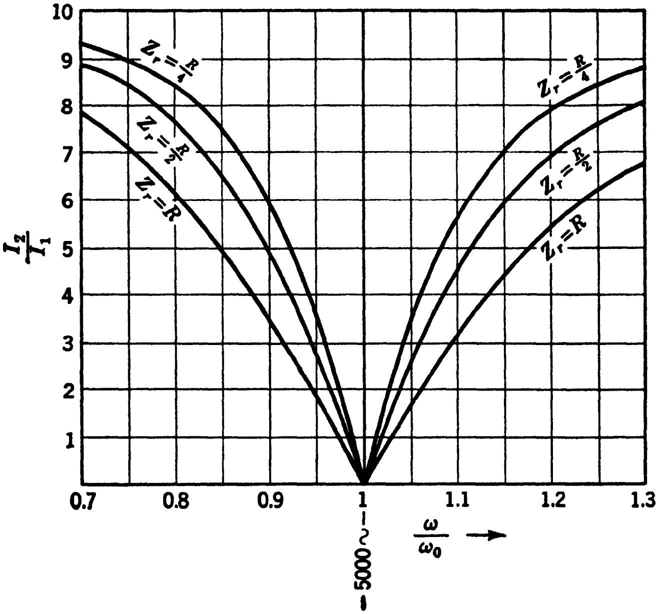


FIG. 9–21. Characteristics of single-frequency rejection filter.

The transmission characteristics of the filter employed in the above example, for three values of terminating impedance, are shown in Fig. 9–21, the above approximation being used. For the case of a terminating impedance of $R/2$,

$$I_2 = I_1 \frac{j\left(\omega L - \dfrac{1}{2\omega C}\right)}{R + j\left(\omega L - \dfrac{1}{2\omega C}\right)} \tag{9–49}$$

or

$$\frac{I_2}{I_1} = \frac{\left(\omega L - \dfrac{1}{2\omega C}\right)}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{2\omega C}\right)^2}} \tag{9–50}$$

The ratio I_2/I_1 is plotted against ω/ω_0 in Fig. 9–21 where ω_0 is the value of $2\pi f$ for infinite rejection.

PROBLEMS

9–1. Calculate the attenuation and phase shift of the T section shown in Fig. 9–22 at $f = 1000$ and $f = 2500$ cycles/sec, employing equation 9–1. The section is terminated in its characteristic impedance at each frequency.

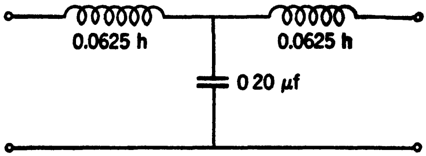


FIG. 9–22 For use in connection with Prob 9–1.

9–2. Calculate the attenuation and phase shift of the π section shown in Fig. 9–23 at $f = 1000$ and $f = 2500$ cycles/sec, employing equation 9–1 The section is terminated in its characteristic impedance at each frequency

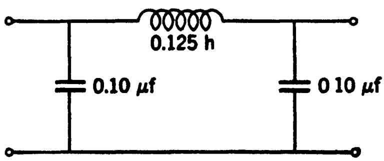


FIG 9–23 For use in connection with Prob 9–2.

9–3. Find the frequency boundaries of the pass band of the filter sections shown in Figs 9–22 and 9–23

9–4. Calculate the characteristic impedance of the T section shown in Fig 9–22 and of the π section shown in Fig 9–23 at 1000 cycles per second, using equations 4–4 and 4–5

9–5. Calculate the characteristic impedance of the T section shown in Fig 9–22 and of the π section shown in Fig 9–23 at 2500 cycles per second, using equations 4–4 and 4–5

9–6. Calculate the characteristic impedance of the T section shown in Fig. 9–22 and of the π section shown in Fig 9–23 at 1000 cycles per second, using the expression $Z_0 = \sqrt{Z_{ss}Z_{so}}$

9–7. Calculate the characteristic impedance of the T section shown in Fig. 9–22 and of the π section shown in Fig 9–23 at 2500 cycles per second, using the expression $Z_0 = \sqrt{Z_{ss}Z_{so}}$

9–8. Design a constant-k low-pass T-section filter with a cut-off frequency of 1500 cycles per second and a characteristic impedance of 600 ohms at zero frequency.

9–9. For the filter of Prob 9–8, calculate and plot α, β, and Z_0 as a function of frequency from 0 to 2500 cycles per second

9–10. What is the cut-off frequency of a low-pass constant-k π-section filter in which the capacitance of each capacitor is 3 0 microfarads and the inductance of

the series coil is 30 millihenrys? What is the characteristic impedance of the section at 796 cycles per second?

9–11. Design a π-section low-pass filter to give a cut-off frequency of 1500 cycles per second and a characteristic impedance of 600 ohms at zero frequency.

9–12. Design a constant-k high-pass T-section filter to have a cut-off frequency of 2000 cycles per second and a characteristic impedance of 80 ohms at infinite frequency.

9–13. For the filter section of Prob. 9–12 calculate and plot α, β, and Z_0 as a function of frequency from 0 to 5000 cycles per second.

9–14. Design a π-section high-pass filter to have a cut-off frequency of 2000 cycles per second and a characteristic impedance of 80 ohms.

9–15. Given two capacitors of 0.75 microfarad each and a coil of 10 millihenrys, determine the cut-off frequency and characteristic impedance at infinite frequency (high pass) and cut-off frequency and characteristic impedance at zero frequency (low pass) for the filter sections that can be constructed from these elements.

9–16. Design a band-pass filter to have a pass band between 100 and 3000 cycles per second and a characteristic impedance of 600 ohms.

9–17. Design a band-pass filter having a pass band between 2000 and 3000 cycles per second. C_2 is to have a value of 3.0 microfarads. Calculate the characteristic impedance.

9–18. Derive the design equations for a stop band constant-k T-section filter.

9–19. Refer to Fig. 9–18. Derive, by the loop or mesh-current method of circuit analysis, the equivalent circuit shown in Fig. 9–20, employing the necessary approximations. Note: For rejection of currents of a certain frequency through Z_r it is only necessary that the current through Z_r be zero at that frequency.

CHAPTER X

M-DERIVED AND COMPOSITE FILTERS

The filters considered in Chapter IX have met two requirements: (1) the pass band has extended over an interval between two specified frequencies while outside of this band there has existed a region of attenuation, and (2) at a certain frequency the filter has been made to have a given characteristic impedance. These filters, however, have two serious faults: (1) the value of Z_0 is not a constant over the pass band and (2) the attenuation offered to frequencies just outside the pass band, yet near to the cut-off frequency, is not sufficient for ordinary use. Methods which are employed to overcome these difficulties will be treated now in an elementary manner. In this development a certain amount of " mathematical experimentation " will be relied upon inasmuch as certain seemingly aimless mathematical maneuvering will be found to lead to desired results.[1] Later an illustrative design of a composite filter will be presented.

81. The *M*-Derived Section. It is clear that any number of similar filter T sections may be connected in tandem to give almost any required attenuation outside the pass band. Since the attenuation curve of the series of sections would have the same shape as that for one section, shown in Figs. 9–8 and 9–9, a large number of sections would be needed in order to obtain a high attenuation near the cut-off frequency. In order to eliminate the necessity of a large number of sections, a section will be derived which will have an attenuation peak at a selected point in the stop band. This point can be placed wherever desired and when a high attenuation near the cut-off frequency is wanted it can be set very near to the cut-off frequency. The section must be such, however, that it may be connected in tandem with other derived sections and, of course, possibly with the prototype itself. This means that the characteristic impedance, Z_0, of the derived type must be the same as that of the prototype.

Consider the T section shown in Fig. 10–1 in which the prototype is represented at (*a*). It is desired to find a section (*b*) such that its series elements will be equal to Z_1 multiplied by a numeric m and its

[1] The student should refer to Guillemin, *Communication Networks*, Vol. 2, Art. 6, p. 323, John Wiley & Sons, New York.

characteristic impedance will be equal to Z_0 of the prototype. Equate the expressions for Z_0 of the two sections and solve for Z_2'.

$$Z_0 = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} = \sqrt{mZ_1Z_2' + \frac{m^2Z_1^2}{4}}$$

Squaring and rearranging

$$mZ_1Z_2' = Z_1Z_2 + \frac{Z_1^2}{4} - \frac{m^2Z_1^2}{4} = Z_1Z_2 + (1 - m^2)\frac{Z_1^2}{4}$$

or

$$Z_2' = \frac{Z_2}{m} + \frac{(1 - m^2)Z_1}{4m} \qquad [10\text{–}1]$$

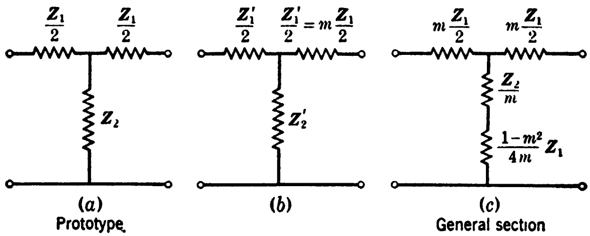


FIG. 10–1. Development of m-derived filter section.

The equation shows that Z_2', as shown in Fig. 10–1c, is made of two parts, one like Z_1 and one like Z_2. Also it indicates that, if the second term on the right of equation 10–1 is always to be like Z_1, then

$$1 - m^2 > 0, \quad \text{or} \quad m < 1$$

An infinite number of m-derived filters exist for which $m < 1$ and all of them have the same Z_0. The nature of this new filter section will now be investigated for two special cases.

It is of course necessary that the cut-off frequencies of m-derived sections be the same as those of the prototype if the two are to be connected in tandem. The frequency boundaries of the m-derived filter sections are obtained by setting

$$\frac{Z_1'}{4Z_2'} = 0$$

and

$$\frac{Z_1'}{4Z_2'} = -1$$

Letting L represent the total series arm inductance of the low-pass prototype (or constant-k) section and C represent the shunt capacitance

of this section, we have for the m-derived low-pass section

$$\frac{Z_1'}{4Z_2'} = \frac{jm\omega L}{\dfrac{-j4}{m\omega C} + \dfrac{(1 - m^2)j\omega L}{m}} = 0$$

or

$$jm\omega L = 0$$

$$\omega = 2\pi f_0 = 0$$

and

$$f_0 = 0 \qquad [9\text{–}11]$$

Also

$$\frac{Z_1'}{4Z_2'} = \frac{jm\omega L}{\dfrac{-j4}{m\omega C} + \dfrac{(1 - m^2)j\omega L}{m}} = -1$$

which yields

$$\omega = 2\pi f_0 = \frac{2}{\sqrt{LC}}$$

and

$$f_0 = \frac{1}{\pi\sqrt{LC}} \qquad [9\text{–}11]$$

Thus it is seen that the frequency boundaries of the pass band of the prototype and m-derived low-pass filter sections are identical. In a similar manner it may be shown that the frequency boundaries of the prototype and m-derived high-pass filter sections are identical. (See Prob. 10–1.)

82. *M*-Derived Low- and High-Pass Filters. For the low-pass prototype filter $Z_1 = j\omega L$ and $Z_2 = -j/(\omega C)$. From the previous article it is established that

$$Z_1' = mZ_1 = jm\omega L$$

and, from equation 10–1,

$$Z_2' = \frac{-j}{m\omega C} + \frac{(1 - m^2)j\omega L}{4m}$$

where the primed letters represent values for the m-derived section.

From these equations the elements for the new section become

$$L_1' = mL \qquad [10\text{–}2]$$

$$L_2' = \frac{1 - m^2}{4m} L \qquad [10\text{–}3]$$

$$C_2' = mC \qquad [10\text{–}4]$$

This newly derived section is shown in Fig. 10–2. At some frequency, which will be designated by f_∞, the shunt arm of this section will be in resonance, since it contains both inductance and capacitance. A resonant condition for the L–C branch simulates a short circuit across the line and thus produces an infinite attenuation. The resonant frequency is given by

$$f_\infty = \frac{1}{2\pi\sqrt{\dfrac{1-m^2}{4m}LmC}} = \frac{1}{\pi\sqrt{(1-m^2)LC}} = \frac{f_0}{\sqrt{1-m^2}} \quad [10\text{–}5]$$

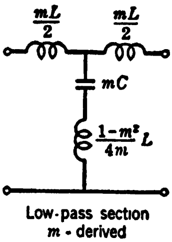


FIG. 10–2. T-section filter.

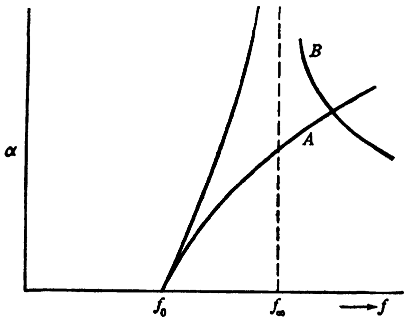


FIG. 10–3. Illustration of attenuation. (A) Low-pass prototype and (B) low-pass m-derived section.

Since m can have any value *between* 0 and 1, it is seen on reference to equation 10–5 that f_∞ must always be higher than $f_0\left(=\dfrac{1}{\pi\sqrt{LC}}\right)$. Thus the frequency for infinite attenuation can be placed wherever desired in the stop band and of course may be set just above the cut-off frequency, thereby greatly increasing the rate of increase of attenuation as the cut-off frequency is passed. Curve A of Fig. 10–3 represents the attenuation of the low-pass prototype, and B represents that of the m-derived section.

Since any section so constructed, regardless of the value of f_∞, has the same value of Z_0, a number of these sections with various f_∞'s can be placed in tandem, producing a high attenuation over the entire stop band.

It is desirable to have an equation giving m in terms of f_0 and f_∞. From equation 10–5,

$$1 - m^2 = \frac{1}{\pi^2 f_\infty^2 LC} = \frac{f_0^2}{f_\infty^2}$$

or

$$m = \sqrt{1 - \frac{f_0^2}{f_\infty^2}} \qquad [10\text{–}6]$$

From the above discussion it is seen that if m is made equal to unity the prototype section results which can be considered as a special case of the m-derived section.

Similar treatment can be applied to the high-pass filter, in which $Z_1 = -j/(\omega C)$ and $Z_2 = j\omega L$. Again using primed letters for the m-derived section

$$Z_1' = mZ_1 = \frac{-jm}{\omega C}$$

$$Z_2' = \frac{j\omega L}{m} - \frac{j(1 - m^2)}{4m\omega C}$$

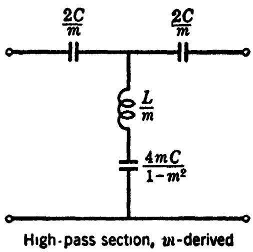


FIG. 10–4. T-section filter.

From these equations the elements for the new section are seen to be

$$C_1' = \frac{C}{m} \qquad [10\text{–}7]$$

$$C_2' = \frac{4mC}{1 - m^2} \qquad [10\text{–}8]$$

$$L_2' = \frac{L}{m}. \qquad [10\text{–}9]$$

This section is represented in Fig. 10–4. The frequency for infinite attenuation (that is, the resonant frequency of the shunt arm) can be calculated from equations 10–8 and 10–9.

$$f_\infty = \frac{1}{2\pi\sqrt{\dfrac{4mC}{1 - m^2} \cdot \dfrac{L}{m}}}$$

$$= \frac{1}{4\pi}\sqrt{\frac{1 - m^2}{LC}} = \sqrt{1 - m^2}\, f_0 \qquad [10\text{–}10]$$

since f_0 for the high-pass filter is $1/(4\pi\sqrt{LC})$. It is seen that f_∞ must always be below the cut-off frequency and thus in the stop band. Again, if m has a very low value, the frequency for infinite attenuation can be placed very near the cut-off frequency. As in the low-pass filter, a number of these m-derived sections with different values of m can be

placed in tandem to produce a high attenuation over the entire stop band.

The equation for m in terms of f_0 and f_∞ can likewise be found for this high-pass filter. From equation 10–10,

$$1 - m^2 = \frac{f_\infty^2}{f_0^2}$$

or

$$m = \sqrt{1 - \frac{f_\infty^2}{f_0^2}} \qquad [10\text{–}11]$$

83. The Characteristic Impedance. In the preceding article it has been shown that a filter can be constructed of a series of m-derived T sections connected in tandem. That they can be connected in tandem is permissible because each section has the same value of Z_0 and the

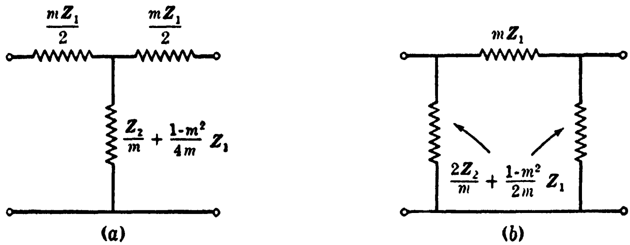


FIG. 10–5. M-derived T and π sections.

same cut-off frequencies. The characteristic impedance has also been shown to be a pure resistance over the pass band although it is variable with frequency. (See Fig. 9–10.) Accordingly it is impossible to match at all frequencies the impedance of the filter by means of a pure resistance alone. As seen in Chapter VII, an impedance match is necessary in order to prevent reflection loss.

The problem will be approached by first constructing a π section from an m-derived T section and then showing that it may be employed as an aid in matching impedances. Begin with the T section of Fig. 10–5a. The same elements when rearranged to form a π section yield the circuit of Fig. 10–5b. The characteristic impedance of the T section is

$$Z_{0T} = \sqrt{Z_1'Z_2' + \frac{Z_1'^2}{4}}$$

$$= \sqrt{mZ_1\left(\frac{Z_2}{m} + \frac{1 - m^2}{4m} Z_1\right) + \frac{m^2 Z_1^2}{4}} = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} \qquad [10\text{–}12]$$

and is independent of the value of m. The characteristic impedance of the π section is given by

$$Z_{0\pi} = \frac{Z_1' Z_2'}{Z_{0T}} \qquad [10\text{–}13]$$

$$= \frac{mZ_1\left[\frac{Z_2}{m} + \frac{(1-m^2)Z_1}{4m}\right]}{\sqrt{Z_1Z_2 + \frac{Z_1^2}{4}}}$$

$$= \frac{Z_1Z_2 + (1-m^2)\frac{Z_1^2}{4}}{\sqrt{Z_1Z_2 + \frac{Z_1^2}{4}}} \qquad [10\text{–}14]$$

and is dependent on the value of m.

For the low-pass filter, equation 10–14 becomes

$$Z_{0\pi} = \frac{\frac{L}{C} - (1-m^2)\frac{\omega^2L^2}{4}}{\sqrt{\frac{L}{C} - \frac{\omega^2L^2}{4}}} = \frac{\sqrt{\frac{L}{C}}\left[1 - (1-m^2)\frac{f^2}{f_0^2}\right]}{\sqrt{1 - \frac{f^2}{f_0^2}}} \qquad [10\text{–}15]$$

and, for the high-pass filter,

$$Z_{0\pi} = \frac{\frac{L}{C} - (1-m^2)\frac{1}{4\omega^2C^2}}{\sqrt{\frac{L}{C} - \frac{1}{4\omega^2C^2}}} = \frac{\sqrt{\frac{L}{C}}\left[1 - (1-m^2)\frac{f_0^2}{f^2}\right]}{\sqrt{1 - \frac{f_0^2}{f^2}}} \qquad [10\text{–}16]$$

If $Z_{0\pi}$ is plotted against frequency, for various values of m, a family of curves as shown in Fig. 10–6 is obtained. It is found that, if $m = 0.6$, $Z_{0\pi}$ is practically constant at the value $\sqrt{L/C}$, for all frequencies less than about $0.85f_0$ for the low-pass filter, whereupon $Z_{0\pi}$ rapidly begins to approach infinity, and for all frequencies greater than about $1.18f_0$ for the high-pass filter. Thus, in order to obtain nearly constant characteristic impedance, such π sections which employ an m of 0.6 are suitable for connecting into lines of constant impedance. If a series of T sections could be made to have the characteristic impedance of the above π section, then our problem would be solved. Consider Fig. 10–7, where an m-derived T section is shown terminated on each side by one-half the above π section. On the right, a load $Z_r = Z_0 = Z_{0\pi} = \sqrt{L/C}$

is connected. The m' used for the T section is not necessarily the same as the m used for the terminating half-sections. Let the impedances looking both ways from points A, B, C, and D be indicated by Z_a, Z_b, etc. as shown. The Z_{0T} between points B and C will be unaffected by a change in the value of m', as noted previously. It must now be proved

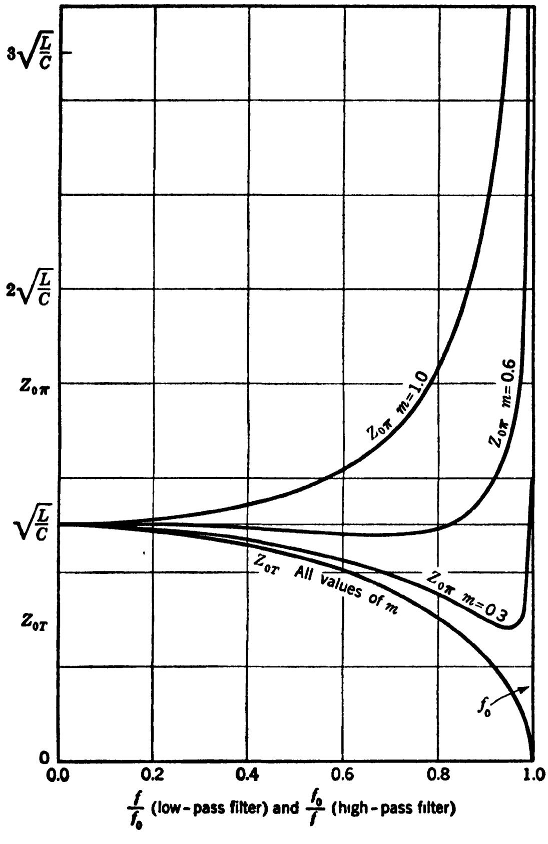


FIG. 10–6. Variation of Z_{0T} and $Z_{0\pi}$ with m over the pass bands.

that the network impedances are matched at all points and thus that, if m is made equal to 0.6, the impedance Z_h, looking into the filter through the terminating half-section, is essentially constant over the pass band.

84. Matching Properties of Terminating Half-Sections. The impedances for Fig. 10–7 are given in terms of Z_1 and Z_2 where

$$Z_1' = mZ_1, \quad Z_2' = \frac{Z_2}{m} + \frac{1 - m^2}{4m} \cdot Z_1, \quad Z_{0\pi} = \frac{Z_1' Z_2'}{Z_{0T}}, \quad Z_1'' = m' Z_1, \quad \text{and}$$

$Z_2'' = \dfrac{Z_2}{m'} + \dfrac{1 - m'^2}{4m'} \cdot Z_1$. It is necessary to show that the impedances are matched at points A, B, C, and D.

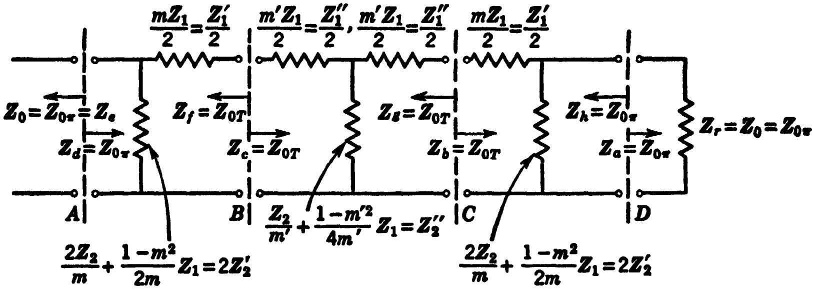


FIG. 10–7. Matching by means of terminating half-sections.

At D, toward the load,

$$Z_a = Z_{0\pi} \tag{10–17}$$

At C, toward the load,

$$Z_b = \frac{Z_1'}{2} + \frac{2Z_2'Z_{0\pi}}{2Z_2' + Z_{0\pi}}$$

$$= \frac{Z_1'Z_2' + \dfrac{Z_1'}{2} \cdot \dfrac{Z_1'Z_2'}{Z_{0T}} + \dfrac{2Z_2'Z_1'Z_2'}{Z_{0T}}}{2Z_2' + \dfrac{Z_1'Z_2'}{Z_{0T}}}$$

$$= \frac{Z_1'Z_2'Z_{0T} + 2Z_2'\left(Z_1'Z_2' + \dfrac{Z_1'^2}{4}\right)}{2Z_2'Z_{0T} + Z_1'Z_2'}$$

$$= \frac{(Z_1'Z_2' + 2Z_2'Z_{0T})Z_{0T}}{2Z_2'Z_{0T} + Z_1'Z_2'} = Z_{0T} \tag{10–18}$$

At B, toward the load,

$$Z_c = \frac{Z_1''}{2} + \frac{Z_2''\left(\dfrac{Z_1''}{2} + Z_{0T}\right)}{Z_2'' + \dfrac{Z_1''}{2} + Z_{0T}}$$

$$= \frac{Z_1''Z_2'' + \dfrac{Z_1''^2}{4} + Z_{0T}\left(\dfrac{Z_1''}{2} + Z_2''\right)}{Z_2'' + \dfrac{Z_1''}{2} + Z_{0T}}$$

$$Z_c = \frac{Z_{0T}\left(\dfrac{Z_1''}{2} + Z_2'' + Z_{0T}\right)}{Z_2'' + \dfrac{Z_1''}{2} + Z_{0T}} = Z_{0T} \qquad [10\text{–}19]$$

At A, toward the load,

$$Z_d = \frac{2Z_2'\left(\dfrac{Z_1'}{2} + Z_{0T}\right)}{2Z_2' + \dfrac{Z_1'}{2} + Z_{0T}} = \frac{Z_1'Z_2' + 2Z_2'\dfrac{Z_1'Z_2'}{Z_{0\pi}}}{2Z_2' + \dfrac{Z_1'}{2} + \dfrac{Z_1'Z_2'}{Z_{0\pi}}}$$

$$= \frac{Z_1'Z_2'\left(Z_{0\pi}\dfrac{Z_1'}{2} + Z_1'Z_2'\right)}{\left(Z_1'Z_2' + \dfrac{Z_1'^2}{4}\right)Z_{0\pi} + Z_1'Z_2'\dfrac{Z_1'}{2}}$$

$$= \frac{\dfrac{Z_{0\pi}}{2Z_2'} + 1}{\dfrac{1}{Z_{0\pi}} + \dfrac{1}{2Z_2'}} = \frac{(Z_{0\pi} + 2Z_2')Z_{0\pi}}{2Z_2' + Z_{0\pi}} = Z_{0\pi} \qquad [10\text{–}20]$$

At A, toward the line,

$$Z_e = Z_{0\pi} = Z_0 \qquad [10\text{–}21]$$

Thus the impedances are matched at point A since $Z_d = Z_e = Z_{0\pi}$. At B, toward the line,

$$Z_f = \frac{Z_1'}{2} + \frac{2Z_2'Z_{0\pi}}{2Z_2' + Z_{0\pi}} = Z_{0T} \quad (\text{See } Z_b) \qquad [10\text{–}22]$$

and the impedances are matched at point B since $Z_c = Z_f = Z_{0T}$. At C, toward the line,

$$Z_g = \frac{Z_1''}{2} + \frac{Z_2''\left(\dfrac{Z_1''}{2} + Z_{0T}\right)}{Z_2'' + \dfrac{Z_1''}{2} + Z_{0T}} = Z_{0T} \quad (\text{See } Z_c) \qquad [10\text{–}23]$$

and the impedances are matched at point C since $Z_b = Z_g = Z_{0T}$.

At D, toward the line,

$$Z_h = \frac{2Z_2'\left(\frac{Z_1'}{2} + Z_{0T}\right)}{2Z_2' + \frac{Z_1'}{2} + Z_{0T}} = Z_{0\pi} \quad \text{(See } Z_d\text{)} \qquad [10\text{–}24]$$

and the impedances are matched at point D since $Z_a = Z_h = Z_{0\pi}$.

If m in the half-section is made equal to 0.6 the impedance looking back into the whole filter from D will be practically constant at the value $Z_{0\pi}$ over the pass band.

The results of this development may be summarized as follows: A constant-k-type T-section filter may be designed having an appropriate cut-off frequency and a desired characteristic impedance. The m-derived sections are then designed on the basis of the T section, m being taken equal to different values, depending on where the frequencies of infinite attenuation are desired. All these sections may be connected in tandem and terminated at each end by one-half a section derived with $m = 0.6$. If such a termination is made, the entire filter will then have a practically constant Z_0 ($= \sqrt{L/C}$ of the prototype) and also the required f_0 and f_∞'s. Such a filter, when connected between lines or pieces of apparatus, will have an impedance which is essentially a constant pure resistance.

85. Illustrative Example of a Low-Pass Filter Design. Let it be required to construct a low-pass filter having $f_0 = 1000$ and $f_\infty = 1200$ cycles/sec, and $Z_0 = 600$ ohms resistance. It is to consist of the prototype, one m-derived section, and the two terminating half-sections.

For the prototype, equations 9–17 and 9–18 will be used.

$$C = \frac{10^6}{\pi \times 1000 \times 600} = 0.530\ \mu\text{f}$$

$$L = \frac{600}{1000\pi} = 0.191 \text{ henry}$$

From equation 10–6,

$$m = \sqrt{1 - \left(\frac{1000}{1200}\right)^2} = 0.553$$

Equations 10–2, 10–3, and 10–4 may be used to find the elements of the m-derived section.

$$L_1' = mL = 0.553 \times 0.191 = 0.1055 \text{ henry}$$

$$L_2' = \frac{1 - m^2}{4m} L = \frac{1 - 0.553^2}{4 \times 0.553} \times 0.191 = 0.060 \text{ henry}$$

$$C_2' = mC = 0.553 \times 0.530 = 0.293\ \mu\text{f}$$

The T section, from which the terminating half-section is obtained, is found by using $m = 0.6$. Let these elements be denoted by L_{1T}, L_{2T}, and C_{2T}.

$$L_{1T} = mL = 0.6 \times 0.191 = 0.1146 \text{ henry}$$

$$L_{2T} = \frac{1 - m^2}{4m} L = \frac{1 - 0.6^2}{4 \times 0.6} \times 0.191 = 0.0509 \text{ henry}$$

$$C_{2T} = mC = 0.6 \times 0.53 = 0.318\ \mu\text{f}$$

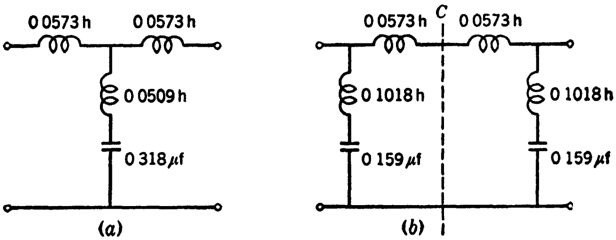


FIG. 10–8. For use in connection with illustrative example of Art. 85.

This m-derived section, which would appear as in Fig. 10–8a, must be rearranged as in Fig. 10–8b in order to be divided at C, thereby making the two terminating half-sections.

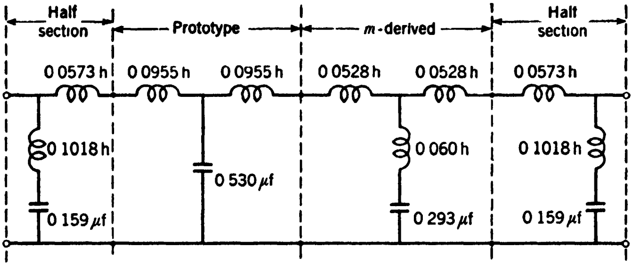


FIG. 10–9 For use in connection with illustrative example of Art. 85.

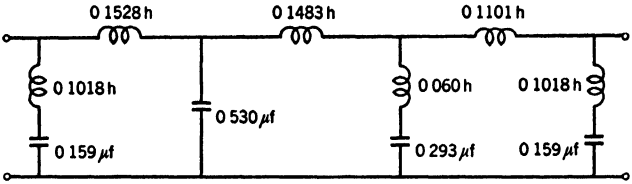


FIG. 10–10. For use in connection with illustrative example of Art. 85.

The complete filter will appear as in Fig. 10–9. In actual construction the elements would be combined as in Fig. 10–10.

85a. α and β of *M*-Derived Filters. The attenuation and phase shift of m-derived filters are obtained also from equation 7–22 although further consideration must be given to the use of this equation because of the fact that Z_1 and Z_2 may be the same type reactances through a large portion of the stop band.

$$\cosh \gamma = \cosh \alpha \cos \beta + j\sinh \alpha \sin \beta = 1 + \frac{Z_1}{2Z_2} = \text{(a real quantity)}$$

Since $\cosh \gamma$ is real, the imaginary term is zero and

$$\sinh \alpha \sin \beta = 0$$

The condition that $\sinh \alpha \sin \beta$ be zero can be met in three ways:

(1) $\sinh \alpha = 0,$	$\alpha = 0,$	$\cosh \alpha = 1,$	$\beta =$ anything
(2) $\sin \beta = 0,$	$\beta = 0,$	$\cos \beta = 1,$	$\alpha =$ anything
(3) $\sin \beta = 0,$	$\beta = \pi,$	$\cos \beta = -1,$	$\alpha =$ anything

It is seen that $\alpha = 0$ only in case (1). Thus case (1) represents the pass band while cases (2) and (3) represent stop bands. Let these conditions be applied to the $\cosh \alpha \cos \beta$ term of $\cosh \gamma$, then

	Range of Values	Phase Shift
(1) $\cosh \gamma = (1) \cos \beta$	$(-1 \text{ to } +1)$	$\beta°$
(2) $\cosh \gamma = \cosh \alpha\ (1)$	$(+1 \text{ to } \infty)$	0°
(3) $\cosh \gamma = \cosh \alpha\ (-1)$	$(-1 \text{ to } -\infty)$	180°

For the m-derived filter, when Z_1/Z_2 is positive $\cosh \gamma$ lies between $+1$ and ∞ and the phase shift is zero whereas when Z_1/Z_2 is negative $\cosh \gamma$ lies between $-\infty$ and $+1$. In this latter range, the phase shift is 180° when in the stop band but varies between 180° and 0° in the pass band.

PROBLEMS

10–1. Show that the frequency boundaries of the m-derived high-pass filter section are the same as those of the high-pass prototype section, namely:

$$f = \frac{1}{4\pi\sqrt{LC}} \quad \text{and} \quad f = \infty$$

10–2. Design an m-derived T-section low-pass filter with $f_0 = 1500$ cycles/sec, $f_\infty = 1600$ cycles/sec, and $Z_0 = 100$ ohms at zero frequency.

10–3. (*a*) Draw the circuit configuration of an m-derived T-section low-pass filter which will yield infinite attenuation at $f_\infty = 1000$ cycles/sec and which, when placed in series with the section shown in Fig. 10–11, will match that section at all frequencies within the pass band. Give the magnitudes of all circuit elements of the m-derived filter section.

(*b*) Determine the characteristic impedance of the m-derived section at $f = 796$ cycles/sec using the expression $Z_0 = \sqrt{Z_{ss}Z_{so}}$.

10-4. Design an m-derived T-section high-pass filter with $f_0 = 1500$ cycles/sec. $f_\infty = 1400$ cycles/sec, and $Z_0 = 80$ ohms at infinite frequency.

10-5. (*a*) Draw the circuit configuration of an m-derived T-section high-pass filter which will yield infinite attenuation at $f_\infty = 500$ cycles/sec and which, when placed in series with the section shown in Fig. 10-12, will match that section at all frequencies within the pass band. Give the magnitudes of all circuit elements of the m-derived filter section.

(*b*) Determine the characteristic impedance of the m-derived section at $f = 1592$ cycles/sec using the expression $Z_0 = \sqrt{Z_{ss}Z_{so}}$.

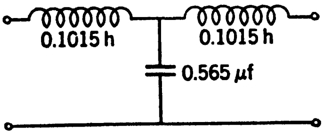


FIG. 10-11. For use in connection with Prob. 10-3.

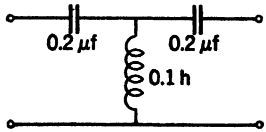


FIG. 10-12. For use in connection with Prob. 10-5.

10-6. Calculate and plot α and β for the filter sections of Probs. 10-2 and 10-4 over the range of frequencies from 0 to 3000 cycles per second.

10-7. Design an m-derived T-section low-pass filter with $f_0 = 10{,}000$ cycles/sec, $f_\infty = 13{,}000$ cycles/sec, and $Z_0 = 70$ ohms at zero frequency.

10-8. Design an m-derived T-section high-pass filter with $f_0 = 10{,}000$ cycles/sec, $f_\infty = 8000$ cycles/sec, and $Z_0 = 70$ ohms at infinite frequency.

10-9. Calculate and plot α and β for the filter sections of Probs. 10-7 and 10-8 over the range of frequencies from 0 to 20,000 cycles per second.

10-10. Design a terminating half-section for a low-pass T-section filter having $f_0 = 1500$ cycles/sec. $Z_0 = 100$ ohms at zero frequency.

10-11. Design a terminating half-section for a low-pass T-section filter having $f_0 = 10{,}000$ cycles/sec. $Z_0 = 70$ ohms at zero frequency.

10-12. Design a low-pass composite filter to work into 600 ohms resistance and to have $f_0 = 1200$ cycles/sec. The filter is to have two terminating half-sections, one prototype, one m-derived section with $f_\infty = 1400$ cycles/sec, and one m-derived section with $f_\infty = 1600$ cycles/sec.

10-13. Design a low-pass composite filter to work into 80 ohms resistance and to have $f_0 = 12{,}000$ cycles/sec. The filter is to have two terminating half-sections, one prototype, and one m-derived section with $f_\infty = 14{,}000$ cycles/sec.

10-14. Design a high-pass filter to work into a 70-ohm resistance and to have $f_0 = 5000$ cycles/sec. The filter is to consist of two terminating half-sections, one prototype, and one m-derived section with $f_\infty = 4000$ cycles/sec.

10-15. Over what per cent of the pass band does the characteristic impedance of the filter of Prob. 10-12 remain within 10 per cent of 600 ohms?

10-16. What is the maximum departure, over the first 95 per cent of the pass band, of $Z_{0\pi}$ from the $\sqrt{L/C}$ value when $m = 0.5$?

10-17. Design a high-pass filter to work into a 100-ohm resistance and to have $f_0 = 18{,}000$ cycles/sec. The filter is to consist of two terminating half-sections, one prototype, one m-derived section with $f_\infty = 16{,}000$ cycles/sec, and one m-derived section with $f_\infty = 12{,}000$ cycles/sec.

CHAPTER XI

IMPEDANCE TRANSFORMATION

In previous chapters the importance of impedance matching has been mentioned. It was shown in Art. 24 that maximum power transfer takes place when a generator of a certain impedance Z_g works into a load whose impedance is the conjugate of Z_g, that is, when $Z_r = \hat{Z}_g$. The material covered in Chapters VI and VII brought out the fact that any mismatch of impedances on a line leads to the existence of reflections, and to a consequent loss of power. It is often necessary to alter the impedance of a line or element in order to match it to a supply line or other source of power, and at the same time either to avoid unnecessary loss or to produce a given amount of loss. The general problem of impedance matching is treated in this chapter in an elementary manner, and some applications of matching to high-frequency transmission are considered. In the general discussion of matching it is immaterial whether conjugate matching or matching for elimination of reflections is meant when the generator and load impedances, or sometimes the load impedance alone, are *pure resistances*, as the two conditions are the same for this case. For matching of general impedances, however, it must be clearly stated which kind of match is required.

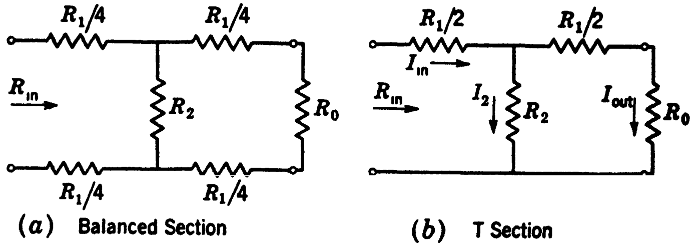


FIG. 11-1. Attenuating sections.

86. Attenuating Sections. As an elementary consideration of impedance matching let the problem be proposed to design an attenuating section, which is commonly referred to as an H pad, such that it will properly match the load resistance into which it must work and at the same time produce a certain decibel loss. Such sections are used to control the power level to those telephones close to the central office

battery so that they will have essentially the same characteristics as those telephones which are farthest removed from the battery. The balanced type of section shown in Fig. 11–1*a*, in which the potentials of corresponding points on either side of the midpoints of R_2 and R_0 are the same, will be rearranged to form the T section of Fig. 11–1*b* for the following analysis.

For the section to match the load resistance R_0 it is only necessary to make $R_{in} = R_0$, thus:

$$R_{in} = R_0 = \frac{R_1}{2} + \frac{\left(\frac{R_1}{2} + R_0\right) R_2}{\frac{R_1}{2} + R_0 + R_2} \qquad [11\text{–}1]$$

Solving for R_0,

$$R_0 = \sqrt{R_1 R_2 + \frac{R_1^2}{4}} \quad \text{(characteristic resistance of the network)} \qquad [11\text{–}2]$$

By means of this equation R_{in} can be made equal to R_0 by simply choosing a resistance value for either R_1 or R_2 and solving for R_2 or R_1 in terms of R_0 and R_1 or R_2 as the case may be. By this process an infinite number of solutions is possible. However, a further restriction has been set, namely, that the attenuation be a specified amount. The attenuation is given by

$$\text{db} = 10 \log \frac{P_{out}}{P_{in}} \qquad [11\text{–}3]$$

where P_{out} = power output and P_{in} = power input. Since $R_{in} = R_0$, we may write

$$n \text{ (= number of db)} = 10 \log \frac{I_{out}^2}{I_{in}^2} = 20 \log \frac{I_{out}}{I_{in}}$$

From this relation

$$\frac{I_{out}}{I_{in}} = 10^{n/20} = k \qquad [11\text{–}4]$$

and, if n is specified, then the ratio I_{out}/I_{in} $(=k)$ is a fixed quantity. From Fig. 11–1*b* it is seen that

$$I_{in} = I_{out} + I_2$$

$$= I_{out} + \frac{I_{out}\left(R_0 + \frac{R_1}{2}\right)}{R_2}$$

$$I_{\text{in}} = I_{\text{out}}\left[1 + \frac{R_0 + \frac{R_1}{2}}{R_2}\right]$$

and

$$\frac{I_{\text{out}}}{I_{\text{in}}} = \frac{R_2}{R_2 + R_0 + \frac{R_1}{2}} = k \qquad [11\text{–}5]$$

Square equation 11–2 to remove the radical; then

$$R_0^2 = R_1R_2 + \frac{R_1^2}{4} \qquad [11\text{–}6]$$

Solve equation 11–5 for R_1, and substitute this value of R_1 into equation 11–6, obtaining for R_2

$$R_2 = 2R_0\frac{k}{1 - k^2} \qquad [11\text{–}7]$$

On substitution of R_2 into equation 11–5 there is obtained for R_1

$$R_1 = 2R_0\frac{1 - k}{1 + k} \qquad [11\text{–}8]$$

Thus R_1 and R_2 are determined in terms of R_0 and the requisite number of decibel loss.

For an attenuating section, k will be less than unity, and R_1 and R_2 will be real quantities. That k will be less than 1 can be shown as follows: For an attenuating section, P_{out} will be less than P_{in} by the amount of the I^2R loss in the section and for R_0 equal to R_{in}, I_{out} likewise will be less than I_{in}. Then $I_{\text{out}}/I_{\text{in}}$ will be less than 1, and, since the log of a number less than 1 is negative, the relation,

$$n = 20\log\frac{I_{\text{out}}}{I_{\text{in}}}$$

will yield a negative value for n. However, it is more convenient to work with positive quantities so n will be taken as positive and k, which is $10^{n/20}$ on the basis presented above will be written as

$$k = 10^{-n/20} = \frac{1}{10^{n/20}}$$

where it is to be remembered that n is positive. k, thus defined, will be less than 1 for all values of n greater than zero.

87. Illustrative Example. An attenuating section is desired that will match a load resistance of 150 ohms and have an attenuation of 6 decibels. What are the required values of R_1 and R_2?

$$R_0 = 150 \text{ ohms}$$

$$n = 6$$

$$k = 10^{-n/20} = 10^{-6/20} = 0.501$$

$$R_1 = 2R_0 \frac{1-k}{1+k} = 2 \times 150 \times \frac{1-0.501}{1+0.501} = 99.73 \text{ ohms}$$

$$R_2 = 2R_0 \frac{k}{1-k^2} = 2 \times 150 \times \frac{0.501}{1-0.501^2} = 201 \text{ ohms}$$

The attenuating section is shown in Fig. 11–2*a*. The balanced type of section is given in Fig. 11–2*b*.

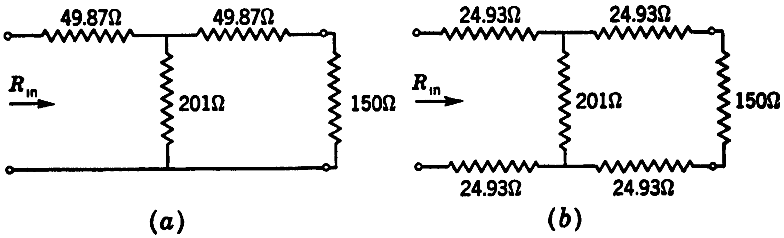


FIG. 11–2. For use in connection with illustrative example of Art. 87.

As a check on the calculations determine whether R_{in} is the required 150 ohms.

$$R_{in} = 49.87 + \frac{199.87 \times 201}{199.87 + 201} = 150 \text{ ohms}$$

88. Impedance Transformation by L Section. In general, a generator or other source of power will not match a load impedance to which it may be connected. Assume that a generator whose internal impedance is Z_g is to be connected to a load impedance Z_r as illustrated in Fig. 11–3*a*. Unless Z_r is the conjugate of Z_g, maximum power will not be transmitted. It is desired to connect into the circuit at *a–b* a simple L-type network which will produce a match of impedances at least at a given frequency. As will be shown, if Z_g and Z_r are pure resistances, the match can be accomplished by the use of pure reactances, and no loss will occur in the auxiliary network. However if Z_g and Z_r are not pure resistances, then a loss may occur in the matching network.

For a reflection match at terminals 1–2,

$$Z_{12} = Z_1 + \frac{Z_3 Z_r}{Z_3 + Z_r} = Z_g \qquad [11\text{–}9]$$

For a similar match at terminals 3–4,

$$Z_{34} = \frac{(Z_1 + Z_g)Z_3}{Z_1 + Z_g + Z_3} = Z_r \qquad [11\text{–}10]$$

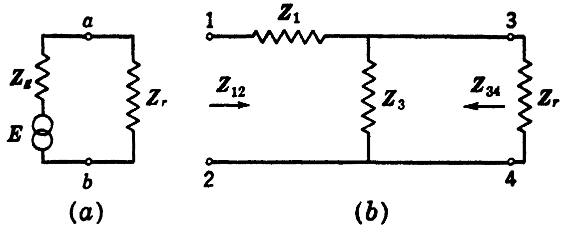


FIG. 11-3. Matching by L section.

From equation 11–9,

$$Z_g Z_3 + Z_g Z_r = Z_1 Z_3 + Z_1 Z_r + Z_3 Z_r \qquad [11\text{–}11]$$

From equation 11–10,

$$Z_1 Z_r + Z_g Z_r + Z_3 Z_r = Z_1 Z_3 + Z_3 Z_g \qquad [11\text{–}12]$$

Subtracting equation 11–12 from equation 11–11 and solving for Z_1 yields

$$Z_1 = \frac{Z_3(Z_g - Z_r)}{Z_r} \qquad [11\text{–}13]$$

Substituting Z_1 from equation 11–13 into equation 11–9 yields an equation involving one unknown, Z_3:

$$Z_3 = \pm Z_r \sqrt{\frac{Z_g}{Z_g - Z_r}} \qquad [11\text{–}14]$$

The value of Z_3 from equation 11–14 is in turn substituted into equation 11–13 to obtain Z_1:

$$Z_1 = \pm Z_r \sqrt{\frac{Z_g}{Z_g - Z_r}} \left(\frac{Z_g - Z_r}{Z_r} \right) = \pm \sqrt{Z_g(Z_g - Z_r)} \qquad [11\text{–}15a]$$

$$= Z_g \frac{1}{\pm \sqrt{\dfrac{Z_g}{Z_g - Z_r}}} \qquad [11\text{–}15b]$$

If Z_g and Z_r are both resistive, and if lossless elements are to be employed in the L section then equations 11–14 and 11–15, show that Z_r must be greater than Z_g. Equation 11–14 may then be written

$$Z_3 = \pm jZ_r\sqrt{\frac{Z_g}{Z_r - Z_g}} \tag{11–16}$$

and equation 11–15a as

$$Z_1 = \pm j\sqrt{Z_g(Z_r - Z_g)} \tag{11–17a}$$

and equation 11–15b as

$$Z_1 = Z_g\frac{1}{\pm j\sqrt{\frac{Z_g}{Z_r - Z_g}}} = \mp jZ_g\frac{1}{\sqrt{\frac{Z_g}{Z_r - Z_g}}} \tag{11–17b}$$

In order to maintain the matching conditions imposed by equations 11–9 and 11–10, the reactances must be of opposite sign, that is, if Z_1 is $+jX_L$, then Z_3 must be $-jX_C$, and vice versa.

When the condition arises that $Z_r < Z_g$ and both are resistive, then the reverse of the configuration of Fig. 11–3*b* may be employed. An interchange of Z_r and Z_g is made and also an interchange of these values in the above formulas. Or, more generalized, Z_1 must be on the side of the lower impedance, Z_r or Z_g.

For $Z_r < Z_g$ and both resistive:

$$Z_3 = \pm jZ_g\sqrt{\frac{Z_r}{Z_g - Z_r}} \tag{11–18}$$

and

$$Z_1 = \pm j\sqrt{Z_r(Z_g - Z_r)} \tag{11–19a}$$

or

$$Z_1 = \mp jZ_r\frac{1}{\sqrt{\frac{Z_r}{Z_g - Z_r}}} \tag{11–19b}$$

When complex impedances are present in either or both Z_g and Z_r, then equations 11–14 and 11–15 together with Fig. 11–3 or the following equations 11–20 and 11–21 together with Fig. 11–4 are used. The proper circuit configuration to use for a given set of impedances Z_g and Z_r can be determined by proper manipulation of the signs of Z_1 and Z_3 and the circuit position of Z_1 with respect to Z_r.

$$Z_3 = \pm Z_g\sqrt{\frac{Z_r}{Z_r - Z_g}} \tag{11–20}$$

and

$$Z_1 = \pm\sqrt{Z_r(Z_r - Z_g)} \qquad [11\text{–}21a]$$

or

$$Z_1 = \mp Z_r \frac{1}{\sqrt{\dfrac{Z_r}{Z_r - Z_g}}} \qquad [11\text{–}21b]$$

The $\pm$ signs before the various expressions for Z_1 and Z_3 would indicate that two solutions and possibly four are possible for any given

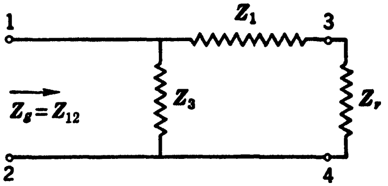


FIG. 11–4. Possible L-section circuit configuration.

matching network. In those cases where Z_g and Z_r are pure resistances, two solutions only are possible since opposite-type reactances must be used for Z_1 and Z_3. In those cases where Z_g and Z_r are complex impedances, in general, only one solution is possible. As is shown in the following illustrative example, one set of impedances for Z_1 and Z_3 would involve the use of negative resistances, and for obvious reasons these would be ruled out. Although two possible circuit configurations are given in Figs. 11–3 and 11–4, it will be found that, in general, only one will yield a decibel gain through matching, and the other will produce a loss. For certain combinations of Z_r and Z_g, no L section can be found which will provide an impedance match and at the same time not introduce a power loss when the resistances required in the matching L section are restricted to positive values.

89. Illustrative Example. (*a*) To design an L section to match $Z_g = 60\,\underline{/0^\circ}$ ohms to $Z_r = 600\,\underline{/0^\circ}$ ohms. What db gain results from inserting the L section? Since $Z_r > Z_g$, use equations 11–16 and 11–17*b*.

From equation 11–16,

$$Z_3 = \pm j600\sqrt{\frac{60}{540}} = \pm j200 \text{ ohms}$$

From equation 11–17b,

$$Z_1 = \mp j60 \frac{1}{\sqrt{\dfrac{60}{540}}} = \mp j180 \text{ ohms}$$

If $\boldsymbol{Z}_1 = +j180$ ohms, then $\boldsymbol{Z}_3 = -j200$ ohms,
or if $\boldsymbol{Z}_1 = -j180$ ohms, then $\boldsymbol{Z}_3 = +j200$ ohms, as shown in Fig. 11–5.

At $\omega = 5 \times 10^7$ radians/sec or $f = 7.96$ megacycles/sec, for Fig. 11–5*a*,

$$L_1 = \frac{180}{5 \times 10^7} = 3.6 \times 10^{-6} \text{ henry} = 3.6 \ \mu\text{h}$$

$$C_3 = \frac{1}{5 \times 10^7 \times 200} = 100 \times 10^{-12} \text{ farad} = 100 \ \mu\mu\text{f}$$

for Fig. 11–5*b*,

$$C_1 = \frac{1}{5 \times 10^7 \times 180} = 111 \times 10^{-12} \text{ farad} = 111 \ \mu\mu\text{f}$$

$$L_3 = \frac{200}{5 \times 10^7} = 4.0 \times 10^{-6} \text{ henry} = 4.0 \ \mu\text{h}$$

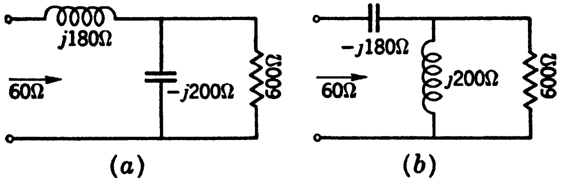


FIG. 11–5. For use in connection with illustrative example of Art. 89*a*.

To determine the change in decibels due to the insertion of the network, assume a generated voltage of E_g. Without the matching section,

$$P_{1\,\text{delivered}} = \left(\frac{E_g}{600 + 60}\right)^2 \times 600 = 0.00138 \ E_g^2 \text{ watts}$$

With the matching section,

$$P_{2\,\text{delivered}} = \left(\frac{E_g}{60 + 60}\right)^2 \times 60 = 0.00417 \ E_g^2 \text{ watts}$$

$$\text{db gain} = 10 \log \frac{P_2}{P_1} = 10 \log 3.025 = 4.8$$

A gain exists since the use of this network results in an increase in transmitted power.

(*b*) To design an L section to match $\boldsymbol{Z}_g = 60 - j30$ ohms to $\boldsymbol{Z}_r = 600 + j300$ ohms for elimination of reflections. Can a gain be obtained by inserting an L section?

The configuration shown in Fig. 11–3*b* will be used first.

$$\boldsymbol{Z}_g = 60 - j30 = 67 \,\underline{/-26.6^\circ} \text{ ohms}$$

$$\boldsymbol{Z}_r = 600 + j300 = 670 \,\underline{/26.6^\circ} \text{ ohms}$$

$$\boldsymbol{Z}_g - \boldsymbol{Z}_r = -540 - j330 = 633 \,\underline{/211.4^\circ}$$

From equation 11–14,

$$Z_3 = \pm Z_r\sqrt{\frac{Z_g}{Z_g - Z_r}} = \pm 670\underline{/26.6^\circ}\sqrt{\frac{67\underline{/-26.6^\circ}}{633\underline{/211.4^\circ}}} = \pm 670\underline{/26\ 6^\circ} \times 0.325\underline{/-119^\circ}$$

$$= \pm 218\underline{/-92.4^\circ} = \pm(-9.14 - j218) = -9.14 - j218 \text{ ohms}$$

$$\text{or} \quad = \quad 9.14 + j218 \text{ ohms}$$

From equation 11–15a,

$$Z_1 = \pm\sqrt{Z_g(Z_g - Z_r)} = \pm\sqrt{67\underline{/-26.6^\circ} \times 633\underline{/211.4^\circ}} = \pm 206\underline{/92.4^\circ}$$

$$= \pm(-8.63 + j206) = -8.63 + j206 \text{ ohms}$$

$$\text{or} \quad = \quad 8\ 63 - j206 \text{ ohms}$$

The network using positive resistance values for Z_1 and Z_3 is shown in Fig. 11–6*a*, and this network produces a decibel gain of 3.2.

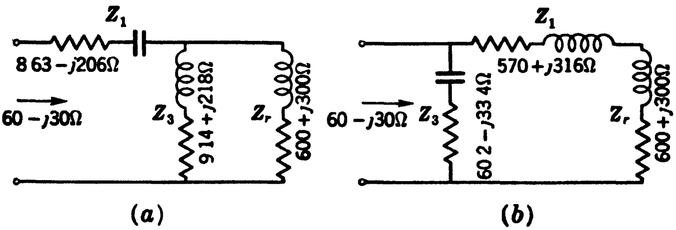


FIG. 11–6 For use in connection with illustrative example of Art 89*b*

When the configuration of Fig. 11–4 is used, Z_3 and Z_1 are obtained by means of equations 11–20 and 11–21a.
From equation 11–20,

$$Z_3 = \pm Z_g\sqrt{\frac{Z_r}{Z_r - Z_g}} = \pm 67\underline{/-26.6^\circ}\sqrt{\frac{670\underline{/26\ 6^\circ}}{633\underline{/31.4^\circ}}} = \pm 68\ 9\underline{/-29^\circ}$$

$$= 60.2 - j33.4 \text{ ohms}$$

$$\text{or} \quad = -60.2 + j33\ 4 \text{ ohms}$$

From equation 11–21a,

$$Z_1 = \pm\sqrt{Z_r(Z_r - Z_g)} = \pm\sqrt{670\underline{/26.6^\circ} \times 633\underline{/31.4^\circ}} = \pm 651\underline{/29^\circ}$$

$$= 570 + j316 \text{ ohms}$$

$$\text{or} \quad = -570 - j316 \text{ ohms}$$

The network using positive resistance values for Z_1 and Z_3 is shown in Fig. 11–6*b*. Although this network produces the necessary matching conditions, it yields, not a gain, but a loss of 11.4 decibels. The student should verify the existence of reflection matching.

90. Tandem Reactive Matching Networks. A study of the networks of Art. 89 will show that, in cases of conjugate matching, if impedances are matched on one side of a reactive coupling, they are also matched on the other side. In this article it will be shown to be generally true that, if a series of pure reactive coupling networks are placed in tandem between two impedances, and if at any junction the impedances looking both ways are matched, that is, conjugates of one another, then the impedances at all other junctions are automatically matched.

Let the diagram of Fig. 11–7 represent a generator or other impedance Z_g, connected through a series of reactive networks A, B, C, and D, to a load impedance Z_r. Suppose that there exists a conjugate impedance match at the junction a between the impedance Z_g and the network. Then maximum possible power is being taken from the generator, as has been shown in Art. 24. If this is the case, there must also be the same power transferred from network A to network B because network A, as

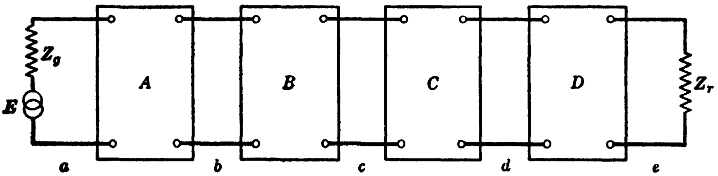


FIG. 11–7. Illustration of reactance theorem.

well as the others, is nondissipative. However, if this same power is being transferred across junction b, then there must exist at b a match of impedances, because, if they were not matched at b, then some adjustment beyond b could be made which would cause matching and thus an increase in power transfer. However, it was assumed that maximum power was already being taken from the generator across junction a, and so an increase in power is impossible. Thus maximum power is already flowing across junction b, and the impedances at b must be matched. This argument may be continued to show that an impedance match occurs at all of the other junctions.

The consequence of the above proof is that, if a line, composed of reactive networks only, is matched to the impedance of a generator or feeder line, then all other junctions on the line are automatically matched. Thus adjustment is only necessary at one point. It should be remembered however that in actual circuits, where resistance cannot be made zero, a slight adjustment may be necessary at several points on the line.

91. Impedance Transformation and Phase Shift Employing Reactive T Sections. Since some Z_g's and Z_r's cannot be matched successfully

with L sections, T sections are often employed. The use of the T section permits the insertion of reactances in the Z_1 and Z_2 arms which will cancel the reactive components of Z_g and Z_r, thus leaving a resistance-to-resistance match which can be accomplished with a reactive L section composed of $Z_1' \mp jX_g$ and Z_3 (for Fig. 11–3) or of $Z_1' \mp jX_r$ and Z_3 (for Fig. 11–4). For reasons which will become apparent presently, however, the T-section analysis will be approached with a view toward allowing an arbitrary phase shift to be introduced at the same time the impedance-matching condition is established.

As an illustration of the above, let it be proposed to design a T section to produce a conjugate match of $Z_g = 60 - j30$ ohms with $Z_r = 600 + j300$ ohms. (See Art. 89*b*.) The T section is shown in Fig. 11-8*a*. First introduce an inductive reactance of $j30$ ohms in the Z_1 arm and a capacitive reactance of $-j300$ ohms in the Z_2 arm to cancel the reactive components of Z_g and Z_r. The problem is now reduced to that of finding an L section to match two resistances as shown in Fig. 11–8*b*. The match of 60 ohms resistance to 600 ohms resistance is given in Art. 89*a*, and the resulting circuits, using these two possible L sections inserted into Fig. 11–8*b*, are shown in Figs. 11–8*c* and 11–8*d*. The units would be combined as shown in Figs. 11–8*e* and 11–8*f*. Power transfer is now a maximum since conjugate impedances are obtained and no loss appears in the matching network. The decibel gain is 5.46 as compared to 3.2 found in Art. 89*b*.

The reactive T section for matching two impedances will be considered as a prelude to the problem of matching resistances and simultaneously providing a desired phase shift. In Fig. 11–8*a* are shown two impedances Z_g and Z_r (now resistances) which are to be matched by the T section composed of Z_1, Z_2, and Z_3. Since various combinations of reactances can be used to effect an impedance match, these reactances will be designated as Z's for the time being.

For a reflection match at terminals 1–2,

$$Z_g = Z_1 + \frac{Z_3(Z_2 + Z_r)}{Z_2 + Z_3 + Z_r} \qquad [11\text{–}22]$$

For a similar match at terminals 3–4,

$$Z_r = Z_2 + \frac{Z_3(Z_1 + Z_g)}{Z_1 + Z_3 + Z_g} \qquad [11\text{–}23]$$

It is evident that there are three impedances (reactances) to be determined, and only two equations are available from which to find them. This means that there is a choice to be made in one of the reactances.

Let Z_1 be known to the extent that certain arbitrary values can be assigned to it, and the problem then reduces to finding expressions for Z_2 and Z_3 in terms of Z_g, Z_r, and Z_1.

From equation 11–22,

$$Z_2Z_g + Z_3Z_g + Z_rZ_g = Z_1Z_2 + Z_1Z_3 + Z_1Z_r + Z_2Z_3 + Z_3Z_r \quad [11\text{–}24]$$

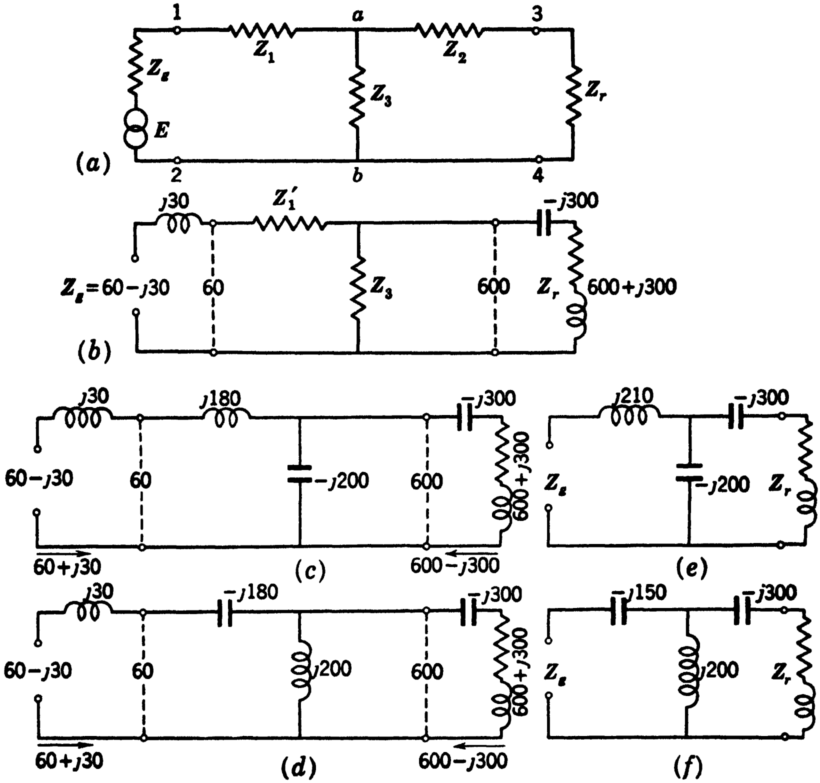


Fig 11–8 Matching by T section All figures represent ohms

From equation 11–23,

$$Z_1Z_r + Z_3Z_r + Z_rZ_g = Z_1Z_2 + Z_2Z_3 + Z_2Z_g + Z_1Z_3 + Z_3Z_g \quad [11\text{–}25]$$

Subtracting equation 11–25 from equation 11–24 and solving for Z_3 yields

$$Z_3 = \frac{Z_1Z_r}{Z_g - Z_r} - \frac{Z_2Z_g}{Z_g - Z_r} \quad [11\text{–}26]$$

Substituting Z_3 from equation 11–26 into equation 11–24 yields

$$Z_2 = \pm\sqrt{\frac{Z_r}{Z_g}}\sqrt{Z_1^2 - Z_g(Z_g - Z_r)} \qquad [11\text{–}27]$$

Inserting the value of Z_2 from equation 11–27 into equation 11–26 obtains

$$Z_3 = \frac{Z_1 Z_r}{Z_g - Z_r} \mp \frac{\sqrt{Z_g Z_r}}{Z_g - Z_r}\sqrt{Z_1^2 - Z_g(Z_g - Z_r)} \qquad [11\text{–}28]$$

A symmetrical network can be obtained by letting $Z_1 = Z_2 = Z/2$; then, from equation 11–27,

$$\frac{Z}{2} = \pm\sqrt{\frac{Z_r}{Z_g}}\sqrt{\frac{Z^2}{4} - Z_g(Z_g - Z_r)}$$

which yields

$$\frac{Z}{2} = \pm\sqrt{-Z_g Z_r} = \pm j\sqrt{Z_g Z_r} \qquad [11\text{–}29]$$

hence $Z/2$ is either inductive or capacitive reactance if Z_g and Z_r are resistive. From equation 11–26,

$$Z_3 = \frac{\frac{Z}{2}Z_r - \frac{Z}{2}Z_g}{Z_g - Z_r} = -\frac{Z}{2} = \mp j\sqrt{Z_g Z_r} \qquad [11\text{–}30]$$

If $Z_1 = Z_2 = Z/2$ are inductive, then Z_3 is capacitive and vice versa.

Illustrative Example. (*a*) Let it be required to design the elements of a reactive symmetrical T section which will correctly match $Z_g = 600\ \underline{/0^\circ}$ ohms and $Z_r = 900\ \underline{/0^\circ}$ ohms at 1000 cycles per second.

From equation 11–29,

$$\frac{Z}{2} = \pm j\sqrt{Z_g Z_r} = \pm j\sqrt{540{,}000} = \pm j735 \text{ ohms}$$

and, from equation 11–30,

$$Z_3 = \mp j\sqrt{Z_g Z_r} = \mp j735 \text{ ohms}$$

For Fig. 11–9*a*,

$$X/2 = \omega L/2 = 735 \text{ ohms}$$

$$L/2 = \frac{735}{2\pi \times 1000} = 0.117 \text{ henry}$$

$$X_3 = 1/(\omega C) = 735 \text{ ohms}$$

$$C = \frac{10^6}{2\pi \times 1000 \times 735} = 0.2165\ \mu\text{f}$$

For Fig. 11–9*b*,

$$X/2 = 1/(\omega C) = 735 \text{ ohms}$$

$$C = 0.2165\ \mu\text{f}$$

$$X_3 = \omega L = 735 \text{ ohms}$$

$$L = 0.117 \text{ henry}$$

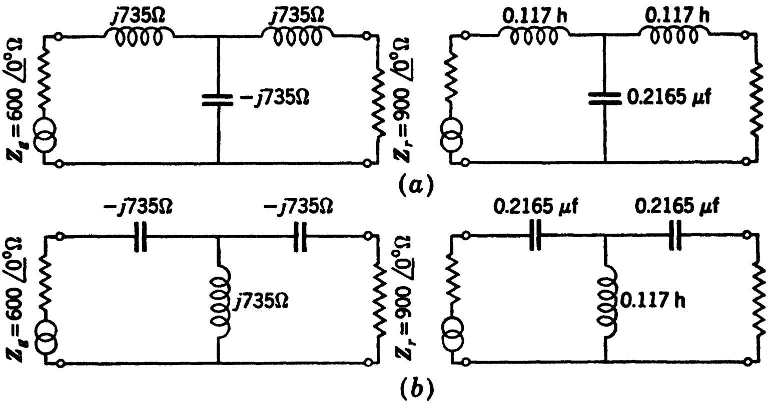


FIG. 11–9. For use in connection with illustrative example *a* of Art. 91.

(*b*) If it desired to form a nonsymmetrical network, then equations 11–27 and 11–28 are used after choosing an appropriate value for Z_1. Let $Z_1 = j542$ ohms. From equation 11–27,

$$Z_2 = \pm\sqrt{\frac{Z_r}{Z_g}}\sqrt{Z_1^2 - Z_g(Z_g - Z_r)}$$

$$= \pm\sqrt{\frac{900}{600}}\sqrt{-542^2 - 600(600 - 900)} = \pm j413 \text{ ohms}$$

From equation 11–28,

$$Z_3 = \frac{Z_1 Z_r}{Z_g - Z_r} \mp \frac{\sqrt{Z_g Z_r}}{Z_g - Z_r}\sqrt{Z_1^2 - Z_g(Z_g - Z_r)}$$

$$= \frac{j542 \times 900}{-300} \mp \frac{\sqrt{600 \times 900}}{-300}\sqrt{-542^2 - 600(600 - 900)}$$

$$= -j1626 \pm j826 = -j800 \quad \text{or} \quad -j2452 \text{ ohms}$$

The $-j800$ ohms is associated with $+j413$ ohms. The two possible sections for the assumed value of $Z_1 = j542$ ohms are shown in Fig. 11–10.

Phase Shift. A type of problem that can be solved conveniently by means of the reactive T section is that of matching a load impedance,

such as an antenna, to a transmission line and simultaneously providing a phase shift of the output current or voltage to phase that antenna correctly with another. For a specified load impedance (antenna) and generator impedance (transmission line) a group of curves may be drawn which will enable the engineer to choose quickly the proper values for Z_1, Z_2, and Z_3 to effect the desired match and phase shift.

Let it be proposed to construct such a set of curves to match $Z_g = 500\underline{/0^\circ}$ ohms to $Z_r = 70\underline{/0^\circ}$ ohms. The impedance Z_1 is again

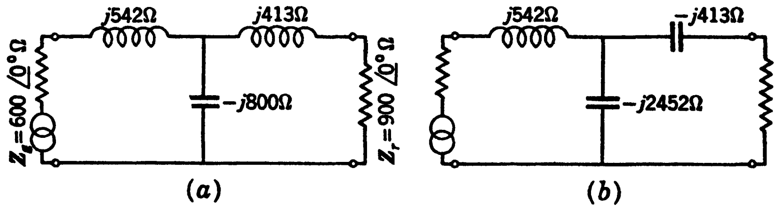


FIG. 11-10. For use in connection with illustrative example *b* of Art. 91.

assumed known; that is, appropriate values will be assigned to it which in the following example will range from 2000 ohms inductive to 2000 ohms capacitive. Z_2 is calculated from

$$Z_2 = \pm\sqrt{\frac{Z_r}{Z_g}}\sqrt{Z_1^2 - Z_g(Z_g - Z_r)} \qquad [11\text{-}27]$$

and Z_3 from

$$Z_3 = \frac{Z_1 Z_r}{Z_g - Z_r} \mp \frac{\sqrt{Z_g Z_r}}{Z_g - Z_r}\sqrt{Z_1^2 - Z_g(Z_g - Z_r)} \qquad [11\text{-}28]$$

It is important to keep in mind that, if the + sign is used in equation 11-27, then the − sign must be used in equation 11-28.

The curves of Fig. 11-11*a* and 11-11*b* give the relationships between Z_1, Z_2, and Z_3. In Fig. 11-11*a*, for example, the curves show that, with $Z_1 = j200$ ohms, Z_2 must be 189 ohms capacitive and Z_3 be 253 ohms inductive to create the match. These illustrative values are given on the diagram *a*.

The phase shift may be determined as follows:

In Fig. 11-8*a* let the current through Z_r be designated I_{out} and the current through Z_g be taken as $1\,\underline{/0^\circ}$ ampere. Then

$$I_{out} = \frac{V_{ab}}{Z_2 + Z_r} = \frac{\dfrac{Z_3(Z_2 + Z_r)}{Z_2 + Z_3 + Z_r} \times 1\underline{/0^\circ}}{Z_2 + Z_r} = \frac{Z_3}{Z_2 + Z_3 + Z_r} \times 1\underline{/0^\circ}$$

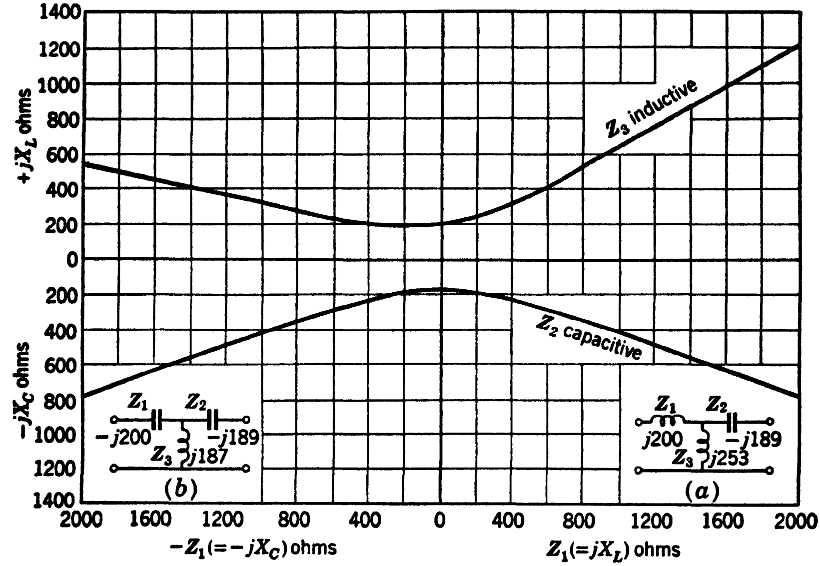


FIG. 11–11*a*. Z_1 versus Z_2 (capacitive) and Z_3 (inductive) for reactive T section used for matching.

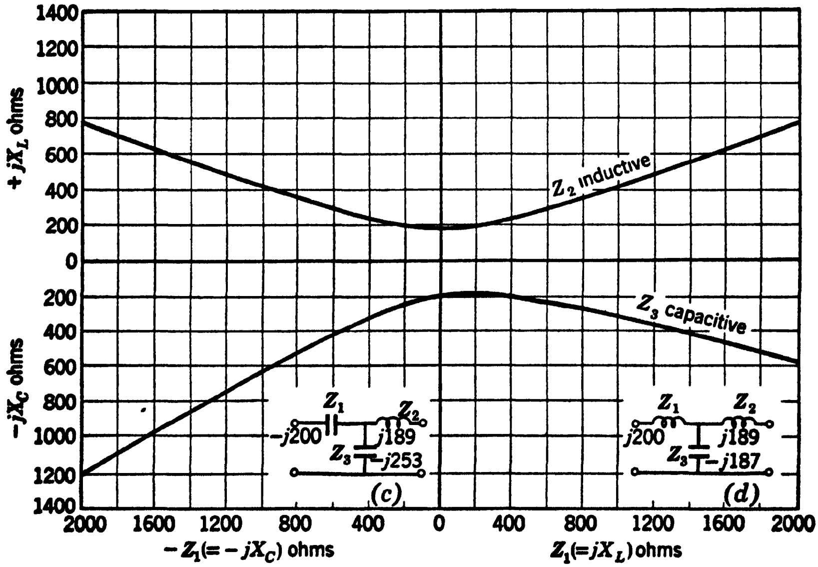


FIG. 11–11*b*. Z_1 versus Z_2 (inductive) and Z_3 (capacitive) for reactive T section used for matching.

$$I_{out} = \frac{|Z_3|\underline{/\alpha^\circ}}{|Z_2 + Z_3 + Z_r|\underline{/\delta^\circ}} \times 1\underline{/0^\circ} = \frac{|Z_3|}{|Z_2 + Z_3 + Z_r|}\underline{/\alpha^\circ - \delta^\circ}$$

$$= \frac{|Z_3|}{|Z_2 + Z_3 + Z_r|}\underline{/\beta^\circ} \qquad [11\text{-}31]$$

where β is the phase shift (in degrees or radians) of the output current of the matching section relative to the input current.

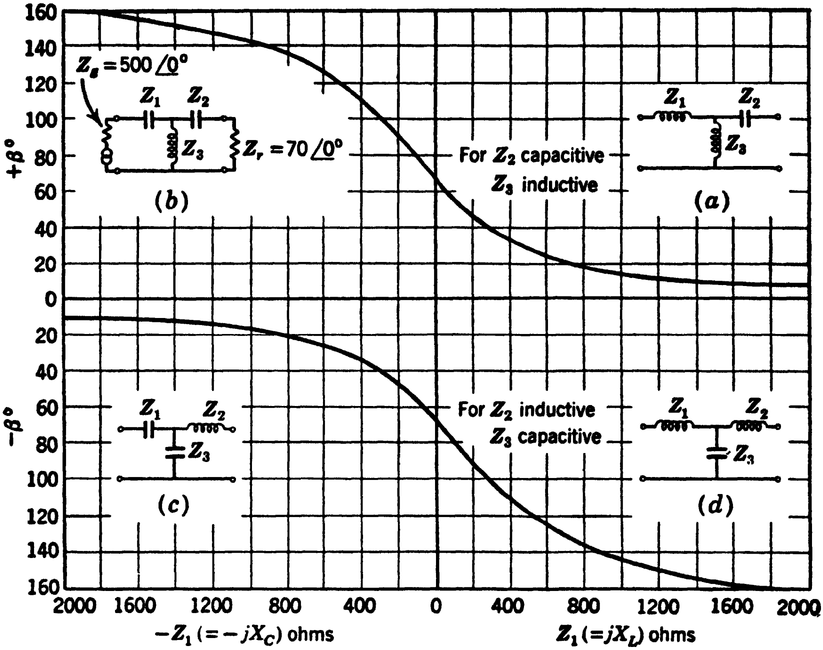


FIG. 11–12. Phase shift β versus Z_1 of reactive T section used for matching.

The curves of Fig. 11–12 show the relationship between Z_1 and β for the case of $Z_g = 500\underline{/0^\circ}$ ohms and $Z_r = 70\underline{/0^\circ}$ ohms. For these particular values of Z_g and Z_r a phase shift of 47.6° is obtained employing $Z_1 = j200$, $Z_2 = -j189$, and $Z_3 = j253$ ohms. If a phase shift other than 47.6° is desired, it is simply necessary to select the appropriate value of Z_1 as determined from Fig. 11–12 and determine Z_2 and Z_3 required to maintain the impedance match between Z_g and Z_r.

92. The High-Frequency Quarter-Wavelength Line as an Impedance Transformer. The characteristic impedance of a T section is given by equation 4–1 and is

$$Z_{0T} = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} \qquad [4\text{-}1]$$

If the total series impedance of the section shown in Fig. 11–8a, namely, $\pm j2\sqrt{Z_gZ_r}$ (from equation 11–29) and the shunt impedance $\mp j\sqrt{Z_gZ_r}$ (from equation 11–30) are substituted into equation 4–1, there is obtained

$$Z_{0T} = \sqrt{Z_gZ_r}$$

and, if the elements of the matching section are reactive, then both Z_g and Z_r are resistive, or

$$Z_{0T} = \sqrt{R_gR_r} \qquad [11\text{–}32]$$

The expression given by equation 11–32 would lead one to suspect that a transmission line, being representable by an equivalent T section, could be used as a coupling circuit to match two resistances. Further support to such a belief is given by the fact that at high frequencies the characteristic impedance is

$$Z_0 \doteq \sqrt{\frac{L}{C}} \quad \text{(a pure resistance)} \qquad [5\text{–}44]$$

This impedance simulates a pure resistance, and the elements of the equivalent circuit of the high-frequency line, being practically pure reactances, cause the loss to be very small. A coaxial line is especially suitable for such matching, provided the necessary unbalance of the coaxial line can be tolerated.

In order to develop the elementary theory of matching by $\lambda/4$ lines (quarter-wavelength lines) it is necessary to refer to equation 6–11, which gives the input impedance of a line in terms of the line parameters and the receiving-end impedance. Let this equation be rewritten, $a + jb$ being substituted for $\sqrt{zy}\,S$.

$$Z_s = Z_0\left[\frac{Z_r \cosh (a + jb) + Z_0 \sinh (a + jb)}{Z_0 \cosh (a + jb) + Z_r \sinh (a + jb)}\right] \qquad [11\text{–}33]$$

For the quarter-wavelength line at high frequency a will be assumed to be zero; then $a = \alpha l = 0$, and β of course is such that $b = \beta l = \pi/2$. If these values are substituted into equation 11–33,

$$Z_s = Z_0\left[\frac{Z_r \cosh j\frac{\pi}{2} + Z_0 \sinh j\frac{\pi}{2}}{Z_0 \cosh j\frac{\pi}{2} + Z_r \sinh j\frac{\pi}{2}}\right]$$

$$= \frac{Z_0^2}{Z_r} \quad \text{(See Appendix III)} \qquad [11\text{–}34]$$

The result can be written

$$Z_s Z_r = Z_0^2 = Z_g Z_r \tag{11-35}$$

Consider that Z_r is a terminating resistance R_r. Then equation 11-35 indicates that, if the line of characteristic impedance Z_0 is terminated in a resistance R_r, the input impedance Z_s will be R_s a resistance because Z_0 is a resistance; and, for a match at the generator, $R_g = R_s$. That is, from equation 11-35,

$$R_g R_r = Z_0^2 \tag{11-36}$$

which is the same as equation 11-32. Thus the line with its terminating impedance will match a generator whose impedance is R_g $(= R_s)$.

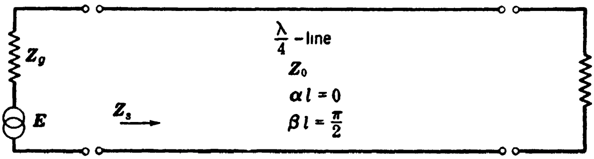


FIG. 11-13. Impedance matching by use of quarter-wavelength line.

(Refer to Fig. 11-13.) If the line is matched at the generator end then it is also matched at the receiver end because of the proof presented in Art. 90. Remember that it has been assumed that the equivalent circuit of the line consists of pure reactances, thus making the remarks of Art. 90 applicable.

It is however a simple matter to show that a match occurs at the receiver end. Consider the line from the receiver end. The terminating impedance will then be R_g, and let the impedance of the line, measured from the receiver end, be written Z_r'. Then equation 11-34 will give Z_r' by merely writing R_g for Z_r and substituting Z_r' for Z_s. This effectively turns the line around, and there is obtained

$$Z_r' = \frac{Z_0^2}{R_g}$$

However, equation 11-36 can be written

$$R_r = \frac{Z_0^2}{R_g} \tag{11-37}$$

which shows that $Z_r' = R_r$, and there exists an impedance match at the receiver end.

93. Illustrative Example. Design a 50-megacycle-per-second quarter-wavelength parallel-wire line to match a resistance of $R_g = 200$ ohms to a load resistance of $R_r = 700$ ohms. The values of L and C are (see equations 1–12 and 1–30)

$$L = 4 \ln \frac{d}{r} \times 10^{-7} \text{ henry/loop meter}$$

$$C = \frac{1}{36 \times 10^9 \ln \frac{d}{r}} \text{ farad/meter}$$

Using equations 5–44 and 11–32

$$Z_0^2 \doteq \frac{L}{C} = R_r R_g = 140{,}000$$

Therefore

$$14{,}400 \ln^2 \frac{d}{r} = 140{,}000$$

whence

$$\ln \frac{d}{r} = 3.12$$

$$\frac{d}{r} = 22.5$$

and

$$d = 22.5r$$

from which it is seen that if No. 0000 copper wire is to be used the spacing should be

$$d = 22.5 \times 0.230 = 5.28 \text{ in.} = 13.15 \text{ cm}$$

At 50 megacycles per second the wavelength will be

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{50 \times 10^6} = 6 \text{ meters}$$

A quarter-wavelength line would have a length of

$$\frac{\lambda}{4} = 1.5 \text{ meters}$$

The characteristic impedance is

$$Z_0 = \sqrt{R_g R_r} = \sqrt{140{,}000}$$

$$= 374 \text{ ohms}$$

The resistance R_g sees $374^2/700 = 200$ ohms and the resistance R_r sees $374^2/200 = 700$ ohms. Thus there is matching from both ends.

94. A Further Property of the Quarter-Wavelength Line. In equation 11–35 there is no restriction on the type of impedances represented by Z_g and Z_r, except that their product must be equal to a positive real number given by L/C. This means that the line under consideration may be used not only for transforming impedance values but also for transforming the character of the impedance. Owing to the inverse relationship of Z_g and Z_r, it is found that a terminating impedance Z_r which may be written $R_r \pm jX_r$ will be transformed into a value of $Z_g = R_g \mp jX_g$. This is shown below. From equation 11–35,

$$
\begin{aligned}
Z_g &= \frac{Z_0^2}{R_r \pm jX_r} \\
&= \frac{Z_0^2(R_r \mp jX_r)}{R_r^2 + X_r^2} \\
&= R_g \mp jX_g
\end{aligned}
$$

where

$$R_g = \frac{Z_0^2 R_r}{R_r^2 + X_r^2} \qquad \text{and} \qquad X_g = \frac{Z_0^2 X_r}{R_r^2 + X_r^2}$$

Thus a capacitive load Z_r can be matched to an inductive generator Z_g, provided the angles of the impedances are the same, or a conjugate match can be effected between a load Z_r and a generator with an impedance Z_g of the same angle. Another way of expressing this is that a load impedance Z_r at the receiver end will appear as some factor times its conjugate at the sending end. If it is desired merely to change an impedance into its conjugate, then a quarter-wavelength line having a value of $Z_0 = \sqrt{R_r^2 + X_r^2}$ should be used; that is, Z_0 $(= \sqrt{L/C})$ must have a value equal to the absolute value of the load impedance. This condition would of course transfer a pure resistance unchanged.

95. The Half-Wavelength Line. The consideration of the use of a quarter-wavelength line in Art. 92 leads to the inquiry as to how a half-wavelength line would serve in such a capacity. For such a line, assuming a negligible value of α, equation 11–33 becomes

$$
\begin{aligned}
Z_s &= Z_0 \cdot \frac{Z_r \cosh j\pi + Z_0 \sinh j\pi}{Z_0 \cosh j\pi + Z_r \sinh j\pi} \\
&= Z_0 \cdot \frac{Z_r(-1) + jZ_0(0)}{Z_0(-1) + jZ_r(0)} \\
&= Z_r
\end{aligned}
\qquad [11\text{–}38]
$$

This simple result means that a half-wavelength line will transform an impedance unchanged. That is, any terminating impedance Z_r will appear at the input of the line as the same impedance. Accordingly the line acts as a one-to-one impedance transformer.

96. The Use of Matching Stubs. Where a high-frequency line is supplying power to a load impedance Z_r which is not equal to the characteristic impedance, there is a consequent loss of power, and it becomes advisable to readjust the impedances so that matching may occur.

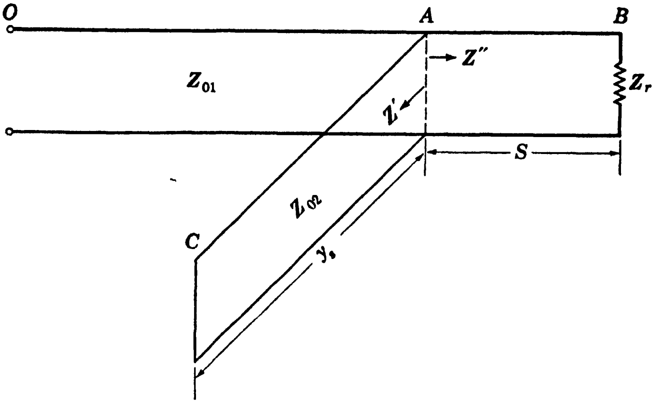


FIG. 11–14. Transmission-line matching by means of short-circuited stub.

A match may be accomplished by shunting across the line near the load a short-circuited line,[1] or stub, approximately one-quarter wavelength long.

A general outline of the method of matching by means of stubs is presented here. (See Fig. 11–14.) At the point A, looking toward the load, there is presented to the line O–A an impedance Z' which is given by equation 6–11 and is in general of the form $r + jx$; that is, it has a resistive and a reactive component. The resistance and reactance are both functions of the distance S to the load. For any particular value of S the impedance Z' can be resolved into an equivalent resistance and reactance in parallel. Call these parallel components R_p and X_p. For some value of S, R_p will be equal to Z_{01} (the characteristic impedance of the line) which in this case is taken as a pure resistance $(= \sqrt{L/C})$. (See equation 5–44.) At point A, X_p will have a certain value which may be either capacitive or inductive and will be of the same

[1] Open-circuit lines, or stubs, may be used also, but they are seldom employed because of the difficulty in supporting two free wires with correct spacing and because of the tendency to radiate at the higher frequencies.

type as the reactance in Z'. In order to eliminate its effect it is only necessary to shunt across the line a reactance of the opposite kind and of the same value. The result will be a resistance R_p in parallel with a parallel-resonant circuit, and the total impedance presented at A to the line O–A will be $R_p (= Z_{01})$, provided S is adjusted to make $R_p = Z_{01}$. Thus the line is matched up to the point A. The shunt reactance can be conveniently constructed of a short-circuited line with a length of about one-quarter wavelength. If the length of the stub is less than $\lambda/4$, the reactance will be inductive, whereas, if the length is greater than $\lambda/4$ but less than $\lambda/2$, the reactance will be capacitive. (See equation 6–47.)

A detailed study of the basic relationships involved will be made starting with equations 5–9 and 5–10 which define the space variations of voltage and current along the line. Slightly rewritten, these equations are

$$V = V_r \cosh (\alpha + j\beta)x + I_r Z_0 \sinh (\alpha + j\beta)x$$

$$I = I_r \cosh (\alpha + j\beta)x + \frac{V_r}{Z_0} \sinh (\alpha + j\beta)x$$

Assume $\alpha = 0$, and recognize that $V_r = I_r Z_r$:

$$V = I_r Z_r \cos \beta x + j I_r Z_0 \sin \beta x \qquad [11\text{–}39]$$

$$I = \frac{1}{Z_0} [I_r Z_0 \cos \beta x + j I_r Z_r \sin \beta x] \qquad [11\text{–}40]$$

For the purposes of this derivation it is desirable to replace equations 11–39 and 11–40 with their exponential equivalents,

$$V = I_r \frac{Z_r + Z_0}{2} \epsilon^{j\beta x} [1 + \mathbf{K}\epsilon^{-j2\beta x}] \qquad [11\text{–}41]$$

$$I = I_r \frac{Z_r + Z_0}{2Z_0} \epsilon^{j\beta x} [1 - \mathbf{K}\epsilon^{-j2\beta x}] \qquad [11\text{–}42]$$

$$\text{where } \mathbf{K} = \frac{Z_r - Z_0}{Z_r + Z_0} = K\underline{/\psi} \quad \text{(called reflection factor)}^2 \qquad [11\text{–}43]$$

Since $\mathbf{K} = K\epsilon^{j\psi}$, equations 11–41 and 11–42 may be written:

$$V = I_r \frac{Z_r + Z_0}{2} \epsilon^{j\beta x} [1 + K\epsilon^{-j(2\beta x - \psi)}] \qquad [11\text{–}44]$$

$$I = I_r \frac{Z_r + Z_0}{2Z_0} \epsilon^{j\beta x} [1 - K\epsilon^{-j(2\beta x - \psi)}] \qquad [11\text{–}45]$$

[2] The magnitude of the reflection factor is the ratio of the magnitude of the reflected wave to the magnitude of the direct wave at the receiving end of the line.

Examination of equation 11–44 shows that for given values of Z_0 and Z_r the voltage coefficient,

$$I_r \frac{Z_r + Z_0}{2}$$

is a fixed quantity and that the operator $\epsilon^{j\beta x}$ produces rotation (for variable x) but does not change the magnitude of the voltage as x varies. The change in voltage along the line is produced by the bracket term. On a per-unit basis where the voltage coefficient is unity, equation 11–44 may be expressed:

$$V' = \epsilon^{j\beta x}\,[1 + K\epsilon^{-j(2\beta x - \psi)}] \qquad [11\text{–}46]$$

A graphical representation of equation 11–46 is given in Fig. 11–15. Since the operator $\epsilon^{j\beta x}$ does not affect the magnitude of V or of V', the

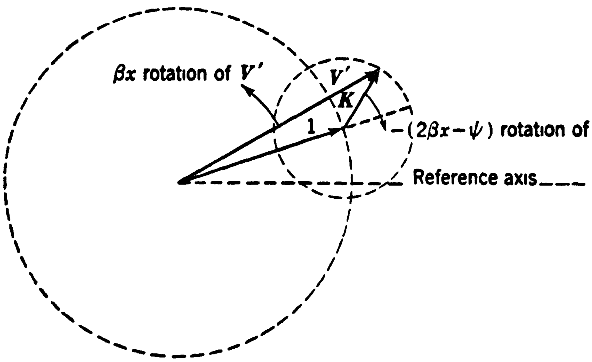


FIG. 11–15. Graphical representation of equation 11–46.

unit vector (1 of Fig. 11–15) may be stopped on the reference axis, and the effect of the $-(2\beta x - \psi)$ rotation of the vector K investigated. In Fig. 11–16 is shown the space variation of V' for the condition of $\psi = 45°$.

From equation 11–46 or the diagram of Fig. 11–16 it is seen that

V is maximum when $2\beta x - \psi = 0$ or where $2\beta x_{\max} = \psi$ [11–47]

V is minimum when $2\beta x - \psi = \pi$ or where $2\beta x_{\min} = \pi + \psi$ [11–48]

It is worth noting that equation 11–45 may be similarly manipulated to show that

I is maximum when $2\beta x - \psi = \pi$ or where $2\beta x_{\max} = \pi + \psi$ [11–49]

I is minimum when $2\beta x - \psi = 0$ or where $2\beta x_{\min} = \psi$ [11–50]

Hence a current minimum occurs at the point of voltage maximum and vice versa.

The important feature to be noted is that the location of the first voltage maximum is defined wholly in terms of Z_0, Z_r, and λ(or f), and hence this point is a convenient reference point from which to work, either analytically or experimentally. From the relationship given in equation 11–47,

$$x_{\max} = \frac{\psi}{2\beta} = \frac{\psi}{4\pi}\lambda \qquad [11\text{–}51]$$

For lines customarily employed for ultrahigh frequencies, Z_0 is essentially resistive, and, if Z_r is resistive, then

ψ is equal to 180° if $Z_r < Z_0$

ψ is equal to 0° if $Z_r > Z_0$

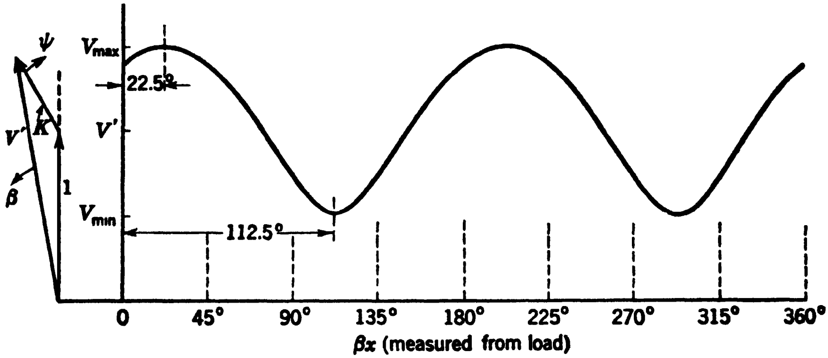


FIG. 11–16. Space variation of V'.

Thus, if $Z_r < Z_0$, the first voltage maximum appears at $\beta x_{\max} = 90°$ or $x_{\max} = \lambda/4$, and, if $Z_r > Z_0$, a voltage maximum appears at $\beta x_{\max} = 0°$ or $x_{\max} = 0$.

A study of Fig. 11–16 will also reveal these facts if ψ is given the values of 180° and 0° respectively. The fact that the K vector revolves at an angular velocity of $2\beta x$ indicates that voltage maxima (and minima) occur every 180° or every half wavelength along the line.

Location of Stubbing Points. In order to determine the location of the stubbing points on the line, that is, points where the equivalent parallel resistance R_p looking toward the load is equal to $Z_0(= Z_{01})$, it is convenient to employ the reciprocal of the equivalent parallel resistance G_x. At these points

$$R_p = Z_{01}, \quad G_x = \frac{1}{R_p}, \quad \text{and} \quad G_x = \frac{1}{Z_{01}} \qquad [11\text{–}52]$$

The admittance of the line at x linear distance or βx angular distance from the load is, by equations 11–44 and 11–45,

$$Y_x = \frac{I_x}{V_x} = \frac{1}{Z_{01}}\left[\frac{1 - K\epsilon^{-j2(\beta x - \beta x_{\max})}}{1 + K\epsilon^{-j2(\beta x - \beta x_{\max})}}\right] \qquad [11\text{–}53]$$

where ψ is equal to $2\beta x_{\max}$. (See equation 11–47.)

A simplification will result from setting $(\beta x - \beta x_{\max})$ equal to Δ, as shown in Fig. 11–17.

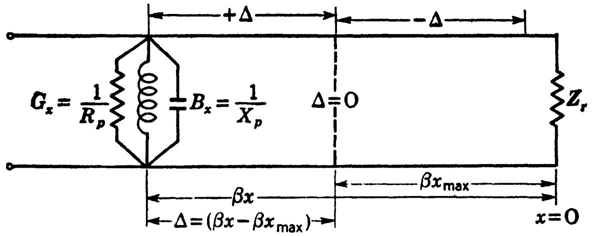


FIG. 11–17. $\Delta \equiv (\beta x - \beta x_{\max})$ is the angular distance between the position of the first voltage maximum and an arbitrarily selected point on the line, βx.

If Δ is used in place of $(\beta x - \beta x_{\max})$ in equation 11–53, an expression for Y_x as a function of the angular distance from the point of first voltage maximum is obtained.

$$Y_x = \frac{1}{Z_{01}}\left[\frac{1 - K\epsilon^{-j2\Delta}}{1 + K\epsilon^{-j2\Delta}}\right]$$

$$= \frac{1}{Z_{01}}\left[\frac{1 - K^2 + K(\epsilon^{j2\Delta} - \epsilon^{-j2\Delta})}{1 + K^2 + K(\epsilon^{j2\Delta} + \epsilon^{-j2\Delta})}\right] \qquad [11\text{–}54]$$

By replacing the exponentials with their trigonometric equivalents

$$Y_x = \frac{1}{Z_{01}}\left[\frac{1 - K^2 + j2K \sin 2\Delta}{1 + K^2 + 2K \cos 2\Delta}\right] \qquad [11\text{–}55]$$

The conductance of the line x distance from the receiver is

$$G_x = \frac{1}{Z_{01}}\left[\frac{1 - K^2}{1 + K^2 + 2K \cos 2\Delta}\right] \qquad [11\text{–}56]$$

and the susceptance of the line is

$$B_x = \frac{1}{Z_{01}}\left[\frac{2K \sin 2\Delta}{1 + K^2 + 2K \cos 2\Delta}\right] \qquad [11\text{–}57]$$

where Δ is angular distance measured from the first voltage maximum.

Proper stubbing points along the line occur only where $G_x = 1/Z_{01}$ since reactive stubs are to be used. At these points the equivalent parallel resistance of the line (looking toward the load Z_r) is equal to Z_{01} which for the ultrahigh frequency line is essentially resistive. From equation 11–56 it is evident that this condition exists when

$$1 - K^2 = 1 + K^2 + 2K \cos 2\Delta \qquad [11\text{–}58]$$

or

$$\cos 2\Delta = -K$$

$$\Delta = \frac{\cos^{-1}(-K)}{2} \qquad [11\text{–}59]$$

Since it is desirable to express Δ using the positive angle K and remembering that the cosine is negative in the second and third quadrants, there are obtained four possible values for Δ in terms of the $(+K)$ angle,

$$\Delta = \pm \frac{180° \pm \cos^{-1} K}{2} \qquad [11\text{–}59a]$$

The expression for Δ locates the correct stubbing positions (in angular measure) from the position of the first voltage maximum. The first set of $\pm$ signs implies that Δ may be measured either toward the generator $(+)$ or toward the receiver $(-)$.

A study of equation 11–59a will show that corresponding stubbing points occur every 180° along the line and that there are two correct stubbing points every half wavelength. Since it is desirable to correlate definitely the correct length of short-circuited stub with any arbitrarily selected stubbing point, it is best to neglect the first set of $\pm$ signs in equation 11–59a, thereby employing only $+$ measurements. If a stubbing point thus found appears to be too far removed from the load $(x = 0)$, a point which is 180° closer to the load may be employed, provided this point is on the actual line, that is, not in the region of $-x$.

The position of the first voltage maximum from the receiver $(x = 0)$ is

$$\beta x_{\max} = \frac{\psi}{2} \quad \text{[See equation 11–47]}$$

Hence the angular distance from the receiver to a correct stubbing point is

$$\beta x_{\text{stub}} = \frac{\psi + (180° \pm \cos^{-1} K)}{2} \qquad [11\text{–}60]$$

and since $$\beta = \frac{2\pi}{\lambda}$$

$$\left.\begin{aligned} x_{stub}\ (= S) &= \frac{\lambda}{4\pi}\,[\psi + (\pi \pm \cos^{-1}K)] \\ &= \frac{\lambda}{720°}\,[\psi + (180° \pm \cos^{-1}K)] \end{aligned}\right\} \quad [11\text{–}61]$$

where $x_{stub}(= S)$ is linear measure from the receiver to the correct stubbing points.

Length of Stub to Be Employed at a Specified Stubbing Point. The susceptance of the line is given by equation 11–57, and at the correct stubbing points

$$\cos 2\Delta = -K \quad \text{and} \quad \sin 2\Delta = \mp\sqrt{1 - K^2}$$

A study of the trigonometric relations involved shows that the − sign before the radical in the expression for sin 2Δ is used together with the + in $\Delta = \dfrac{180° \pm \cos^{-1}K}{2}$ and the + sign before the radical is used with the − in the expression for Δ.

Since the susceptance of the stub (B_s) must be the negative of the susceptance of the line (B_x),

$$\begin{aligned} B_s &= -\frac{1}{Z_{01}}\left[\frac{2K(\mp\sqrt{1 - K^2})}{1 - K^2}\right] \\ &= \frac{1}{Z_{01}}\left[\frac{\pm 2K}{\sqrt{1 - K^2}}\right] \end{aligned} \quad [11\text{–}62]$$

For a short-circuited stub of length y_s whose characteristic impedance is Z_{02} (see Fig. 11–14)

$$Z'' = jZ_{02} \tan B = jZ_{02} \tan \beta y_s = jX'' \quad [11\text{–}63]$$

$$B_s = -\frac{1}{X''} = -\frac{1}{Z_{02} \tan \beta y_s} = \frac{1}{Z_{01}} \cdot \frac{\pm 2K}{\sqrt{1 - K^2}} \quad [11\text{–}64]$$

and

$$\tan \beta y_s = \frac{Z_{01}}{Z_{02}} \cdot \frac{\sqrt{1 - K^2}}{\mp 2K} \quad [11\text{–}65]$$

If the characteristic impedance of the stub is the same as that of the line, then

$$\tan \beta y_s = \frac{\sqrt{1-K^2}}{\mp 2K} \qquad [11\text{–}66]$$

or

$$\beta y_s = \tan^{-1} \frac{\sqrt{1-K^2}}{\mp 2K} \qquad [11\text{–}67]$$

and

$$y_s = \frac{\lambda}{360°} \tan^{-1} \frac{\sqrt{1-K^2}}{\mp 2K} \qquad [11\text{–}68]$$

Care must be exercised in the correct use of the plus–minus signs in equations 11–59a through 11–68. If the first set of $\pm$ signs in equation 11–59a is neglected, then only positive values of Δ are encountered and

$$\Delta_1 = \frac{180° + \cos^{-1} K}{2} \qquad [11\text{–}59b]$$

$$\Delta_2 = \frac{180° - \cos^{-1} K}{2} \qquad [11\text{–}59c]$$

If Δ_1 is selected then

$$\beta y_{s1} = \tan^{-1} \frac{\sqrt{1-K^2}}{-2K} \qquad [11\text{–}69]$$

If Δ_2 is selected then

$$\beta y_{s2} = \tan^{-1} \frac{\sqrt{1-K^2}}{2K} \qquad [11\text{–}70]$$

If a plus sign is used in determining either Δ or βy_s, then a minus sign must be used in determining the other. In the evaluation of βy_s the use of the minus sign indicates that the angular length in question is in the second quadrant, that is, greater than one-quarter wavelength.

The stubbing points, measured from the load, may be expressed as

$$\beta x_{\text{stub }1} = \frac{\psi}{2} + \Delta_1 \qquad [11\text{–}60a]$$

and

$$\beta x_{\text{stub }2} = \frac{\psi}{2} + \Delta_2 \qquad [11\text{–}60b]$$

in angular measure, and as

$$S_1 = \frac{\lambda}{360°}\left(\frac{\psi}{2} + \Delta_1\right) \qquad [11\text{–}61a]$$

and

$$S_2 = \frac{\lambda}{360°}\left(\frac{\psi}{2} + \Delta_2\right) \qquad [11\text{–}61b]$$

in linear measure.

97. Illustrative Example. Let it be required to design a matching stub and to determine the point of contact in order to match a line whose characteristic impedance is $Z_{01} = 600 \underline{/0^\circ}$ ohms to a load of $Z_r = 200 \underline{/0^\circ}$ ohms. The characteristic impedance of the stub is to be the same as that of the line.

$$Z_{01} = Z_0 = 600 \text{ ohms}$$

$$Z_r = 200 \text{ ohms}$$

$$K = \frac{200 - 600}{200 + 600} = -0.5 = 0.5 \underline{/180^\circ}$$

$$\Delta_1 = \frac{180^\circ + \cos^{-1} 0.5}{2} = 120^\circ \quad \text{or} \quad \Delta_2 = \frac{180^\circ - \cos^{-1} 0.5}{2} = 60^\circ$$

$$\beta x_{\text{stub 1}} = 90^\circ + 120^\circ = 210^\circ \text{ or } 30^\circ \qquad \beta x_{\text{stub 2}} = 90^\circ + 60^\circ = 150^\circ$$

$$S_1 = 0.0833\lambda \qquad S_2 = 0.417\lambda$$

$$\beta y_{s1} = \tan^{-1} \frac{\sqrt{1 - 0.5^2}}{-2 \times 0.5} = 139.1^\circ \qquad \beta y_{s2} = \tan^{-1} \frac{\sqrt{1 - 0.5^2}}{2 \times 0.5} = 40.9^\circ$$

$$y_{s1} = 0.386\lambda \qquad y_{s2} = 0.1135\lambda$$

Let Z_r be changed from a value less than Z_{01} to a value greater than Z_{01}.

$$Z_{01} = Z_0 = 600 \text{ ohms}$$

$$Z_r = 1800 \text{ ohms}$$

$$K = \frac{1800 - 600}{1800 + 600} = 0.5 = 0.5 \underline{/0^\circ}$$

$$\Delta_1 = \frac{180^\circ + \cos^{-1} 0.5}{2} = 120^\circ \quad \text{or} \quad \Delta_2 = \frac{180^\circ - \cos^{-1} 0.5}{2} = 60^\circ$$

$$\beta x_{\text{stub 1}} = 0^\circ + 120^\circ = 120^\circ \qquad \beta x_{\text{stub 2}} = 0^\circ + 60^\circ = 60^\circ$$

$$S_1 = 0.333\lambda \qquad S_2 = 0.167\lambda$$

$$\beta y_{s1} = \tan^{-1} \frac{\sqrt{1 - 0.5^2}}{-2 \times 0.5} = 139.1^\circ \qquad \beta y_{s2} = \tan^{-1} \frac{\sqrt{1 - 0.5^2}}{2 \times 0.5} = 40.9^\circ$$

$$y_{s1} = 0.386\lambda \qquad y_{s2} = 0.1135\lambda$$

Let Z_r be changed from a purely resistive value to $Z_r = 150 + j150$ ohms.

$$Z_{01} = 600 \text{ ohms}$$

$$Z_r = 150 + j150 \text{ ohms}$$

$$K = \frac{150 + j150 - 600}{150 + j150 + 600} = 0.62 \underline{/150.23^\circ}$$

$$\Delta_1 = \frac{180^\circ + \cos^{-1} 0.62}{2} = 115.85^\circ \quad \text{or} \quad \Delta_2 = \frac{180^\circ - \cos^{-1} 0.62}{2} = 64.15^\circ$$

$$\beta x_{\text{stub }1} = 75.12° + 115.85° = 190.97° \text{ or } 10.97°$$

$$S_1 = 0.0304\lambda$$

$$\beta y_{s1} = \tan^{-1}\frac{\sqrt{1-0.62^2}}{-2\times 0.62} = 147.7°$$

$$y_{s1} = 0.410\lambda$$

$$\beta x_{\text{stub }2} = 75.12° + 64.15° = 139.27°$$

$$S_2 = 0.387\lambda$$

$$\beta y_{s2} = \tan^{-1}\frac{\sqrt{1-0.62^2}}{2\times 0.62} = 32.3°$$

$$y_{s2} = 0.0897\lambda$$

An extension of this illustrative example is shown in Fig. 11–18 which shows the variation of the ratios V/V_r and I/I_r over the first 360° of

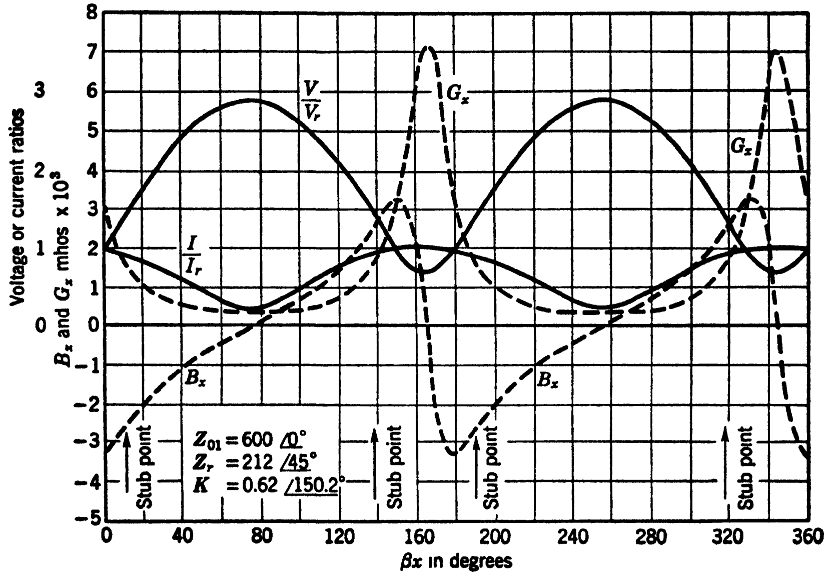


FIG. 11–18. Variation of V/V_r, I/I_r, B_x, and G_x over one wavelength.

line from the position of $x = 0$. Two voltage maxima and two voltage minima are present and, as previously stated, current minima occur at points of voltage maxima and vice versa. Also shown are the variations of B_x and G_x, line susceptance and line conductance. Clearly shown is the fact that similar stubbing points (capacitive stubs) occur at 11° and 191°. In the above example the point at 11° was obtained by moving from the 191° position, as found using the first plus sign in equation 11–59a, to a point 180° closer to the load. Use of the first minus sign in equation 11–59a will yield the 11° point. An inductive stub is placed at the 139.3° point, and, as the curves show, a similar

stub could be placed at the 319.3° point if desired. Examination of the curves show that in 360° there are eight points that yield the correct value of B_x to effect a match, but only four points simultaneously satisfy the second requirement of $G_x = 1/Z_{01}$. For the curves

$$G_x = \frac{1}{Z_{01}}\left[\frac{1 - K^2}{1 + K^2 + 2K\cos 2\Delta}\right] \qquad [11\text{–}56]$$

$$B_x = \frac{1}{Z_{01}}\left[\frac{2K\sin 2\Delta}{1 + K^2 + 2K\cos 2\Delta}\right] \qquad [11\text{–}57]$$

where $2\Delta = 2\beta x - \psi$,

$$\frac{V}{V_r} = \cos\beta x + j\frac{Z_0}{Z_r}\sin\beta x \qquad [11\text{–}71]$$

$$\left|\frac{V}{V_r}\right| = \sqrt{(\cos\beta x + 2\sin\beta x)^2 + (2\sin\beta x)^2} \qquad [11\text{–}72]$$

$$\frac{I}{I_r} = \cos\beta x + j\frac{Z_r}{Z_0}\sin\beta x \qquad [11\text{–}73]$$

$$\left|\frac{I}{I_r}\right| = \sqrt{(\cos\beta x - \tfrac{1}{4}\sin\beta x)^2 + (\tfrac{1}{4}\sin\beta x)^2} \qquad [11\text{–}74]$$

The results of the third illustrative example are given in Fig. 11–19. When stub 1 is used it is seen that the standing wave is virtually eliminated from the line whereas when stub 2 is used the standing wave exists for 0.387λ.

Determination of $\mathbf{Z}_r$, *Stubbing Points and Length of Stubs from Experimental Data.* In those cases where Z_r of the load is unknown, it may be determined from experimental data giving V_{max}/V_{min} and the distance from the load to the first voltage maximum. It is also required that the Z_0 of the line and the transmission frequency be known. From equations 11–41, 11–47, and 11–48 the ratio V_{max}/V_{min} may be written,

$$N = \frac{V_{max}}{V_{min}} = \frac{1 + K\epsilon^{-j(2\beta x_{max} - \psi)}}{1 + K\epsilon^{-j(2\beta x_{min} - \psi)}} = \frac{1 + K\epsilon^{-j0}}{1 + K\epsilon^{-j\pi}}$$

$$= \frac{1 + K}{1 - K} \qquad [11\text{–}75]$$

from which

$$K = \frac{N - 1}{N + 1} \qquad [11\text{–}76]$$

and the associated angle ψ is found from equation 11–47,

$$\psi = 2\beta x_{\max}$$
$$= 2\frac{360°}{\lambda}x_{\max}$$

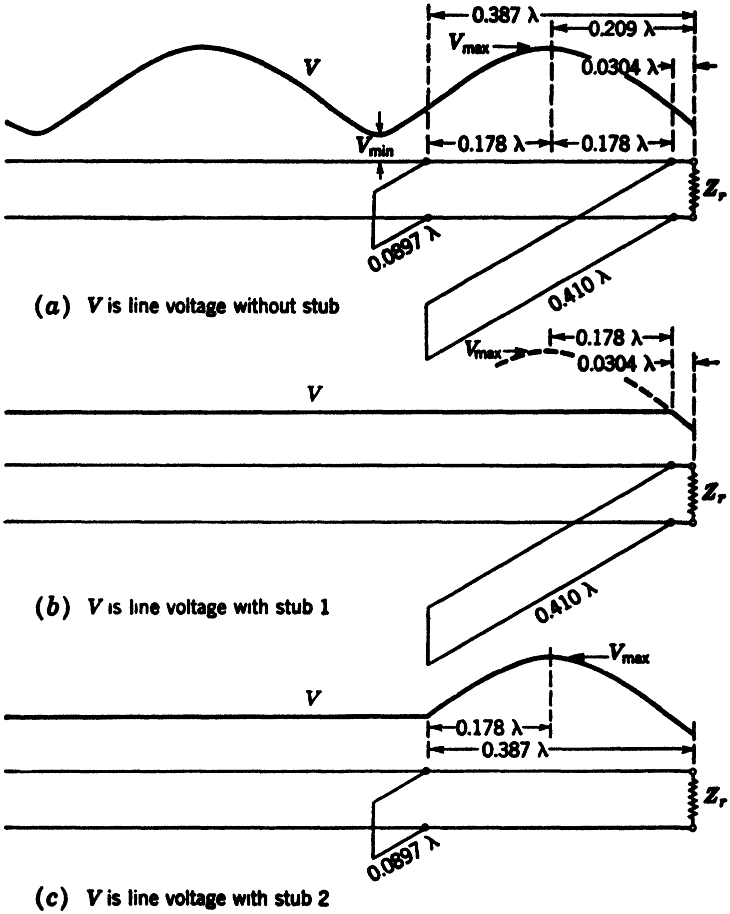


FIG. 11–19. Matching with short-circuited stubs. See Art. 97.

where λ and $x_{\max}$ are linear measure in the same units. From equation 11–43,

$$Z_r = Z_0 \frac{1 + K\underline{/\psi}}{1 - K\underline{/\psi}} \qquad [11\text{–}77]$$

The location of the stubbing points and the length of the stubs may

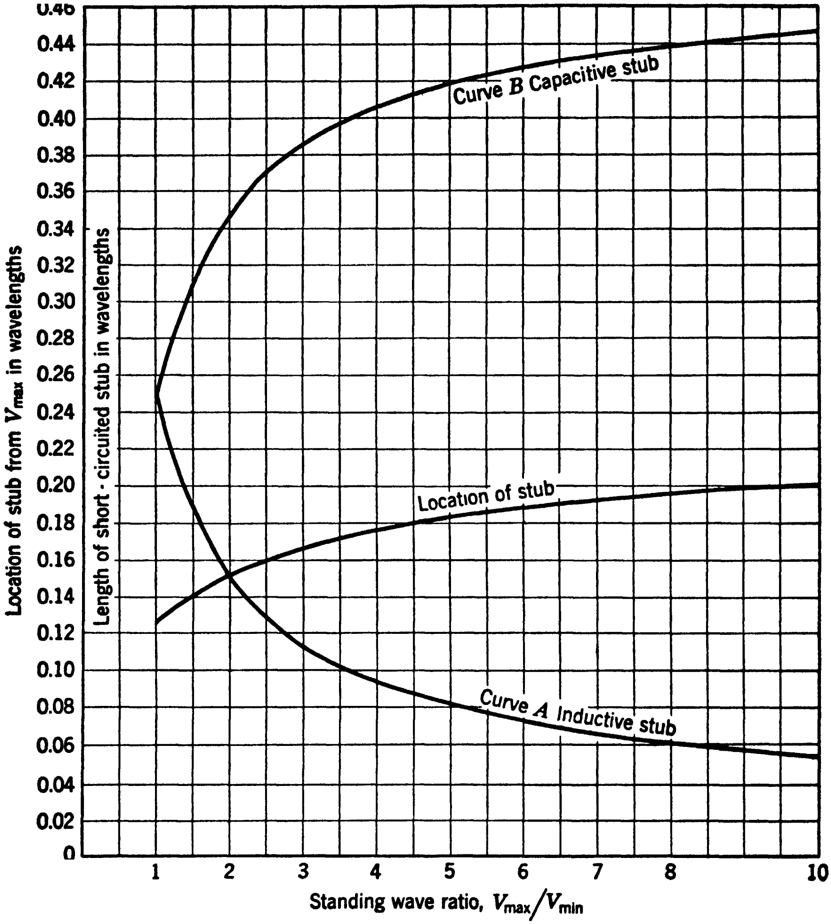


FIG. 11–20. Determination of location and length of matching stub.

be found from the above value of N by writing the equations for Δ and βy_s as

$$\Delta = \pm \frac{180° \pm \cos^{-1}\left(\dfrac{N-1}{N+1}\right)}{2} \qquad [11\text{–}59d]$$

$$\beta y_s = \tan^{-1} \frac{\sqrt{1 - \left(\dfrac{N-1}{N+1}\right)^2}}{\mp 2\left(\dfrac{N-1}{N+1}\right)} = \tan^{-1} \frac{\sqrt{N}}{\mp(N-1)} \qquad [11\text{–}67a]$$

The curves of Fig. 11–20 summarize the results of this article by giving the location of the matching stub with reference to the point of V_{max} and the length of the short-circuited stub. Curve A is for inductive stubs and the location is measured from V_{max} *toward the generator.* Curve B is for capacitive stubs and measurement is made from V_{max} *toward the load.*

The student should check the illustrative examples of Art. 97 with the curves of Fig. 11–20

98. Graphical Treatment of Matching. A graphical construction for impedance matching on high-frequency lines which may be readily set up and which aids in the better understanding of impedance matching as treated in the preceding pages will be presented. The construction is based on the fact that the input admittance of a line having

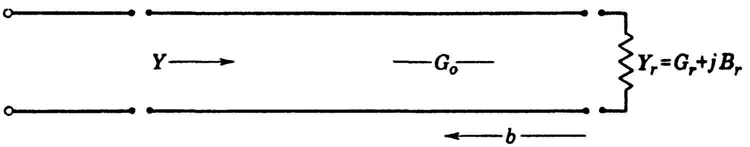


FIG. 11–21. Short section of line.

negligible attenuation plots as a circle when G and B are used as coordinates. Points on this circle represent admittance at any given distance from the receiving end.

A short section of line having negligible attenuation is represented in Fig. 11–21. Let its characteristic admittance be Y_0 ($= 1/Z_0$) and the terminating admittance be $Y_r = G_r + jB_r$. The characteristic admittance of the section will be assumed to be a pure conductance $Y_0 = G_0 = 1/R_0 = \sqrt{C/L}$.

The input impedance of this line of negligible attenuation is found from equation 6–11 and is

$$Z = Z_0 \cdot \frac{Z_r \cos b + jZ_0 \sin b}{Z_0 \cos b + jZ_r \sin b}$$

Expressed in terms of admittances, this equation becomes

$$\frac{1}{Y} = \frac{1}{Y_0} \cdot \frac{\dfrac{1}{Y_r} \cos b + j \dfrac{1}{Y_0} \sin b}{\dfrac{1}{Y_0} \cos b + j \dfrac{1}{Y_r} \sin b}$$

$$= \frac{1}{G_0} \cdot \frac{G_0 \cos b + jY_r \sin b}{Y_r \cos b + jG_0 \sin b}$$

Inversion of the latter equation gives

$$Y = G + jB = G_0 \frac{Y_r \cos b + jG_0 \sin b}{G_0 \cos b + jY_r \sin b} \qquad [11\text{–}78]$$

or

$$G + jB = G_0 \cdot \frac{(G_r + jB_r) \cos b + jG_0 \sin b}{G_0 \cos b + j(G_r + jB_r) \sin b}$$

On cross-multiplication and then division of each term by cos b, there results

$$G_0G + jBG_0 - (GB_r + G_rB) \tan b + j(GG_r - BB_r) \tan b$$
$$= G_0G_r + jG_0B_r + jG_0^2 \tan b$$

By separating real and quadrature terms, two equations are obtained:

$$G_0G - (GB_r + G_rB) \tan b - G_0G_r = 0 \qquad [11\text{–}79]$$

$$BG_0 + (GG_r - BB_r) \tan b - G_0B_r - G_0^2 \tan b = 0 \qquad [11\text{–}80]$$

Elimination of tan b from these equations gives

$$G^2 - G\frac{(G_r^2 + B_r^2 + G_0^2)}{G_r} + B^2 = -G_0^2 \qquad [11\text{–}81]$$

Completing the square involving G and rearranging, there is obtained

$$\left[G - \frac{G_r^2 + B_r^2 + G_0^2}{2G_r}\right]^2 + B^2 = \left[\left(\frac{G_r^2 + B_r^2 + G_0^2}{2G_r}\right)^2 - G_0^2\right] \qquad [11\text{–}82]$$

Equation 11–82 is the equation of a circle, the center of which is at

$$\left.\begin{aligned} B &= 0 \\ G &= \frac{G_r^2 + B_r^2 + G_0^2}{2G_r} \end{aligned}\right\} \qquad [11\text{–}83]$$

and whose radius is

$$r = \sqrt{\left(\frac{G_r^2 + B_r^2 + G_0^2}{2G_r}\right)^2 - G_0^2} \qquad [11\text{–}84]$$

A circle based on the above specifications is shown in Fig 11–22 [3]

The circumference of this circle will contain all the possible values of the input admittance Y, including of course Y_r, obtained when $b = 0$. As b increases, the value of Y will proceed around the circle

[3] With the definition of $Y = G + jB$ which has been adopted in the derivation, an inductive susceptance is a negative quantity and a capacitive susceptance is a positive quantity in the equations, and in depicting these quantities on a G–B plot an inductive susceptance is plotted below the zero ordinate and a capacitive susceptance is plotted above.

from point a in a clockwise direction and will become a pure conductance at the point c, where Y is written as G. If G is equal to the characteristic admittance of a line to the left of the section shown in Fig.

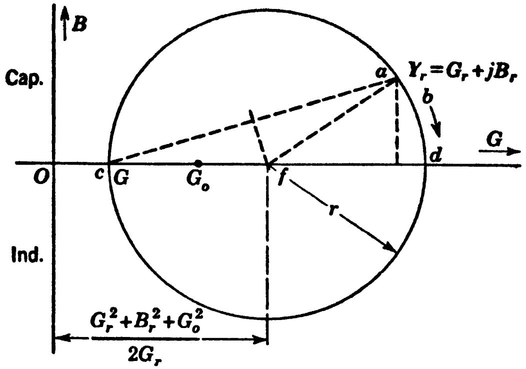


FIG. 11–22. Circle diagram for impedance matching.

11–21, then an impedance match is obtained. This situation is shown in Fig. 11–23.

The problem is usually presented by having given a line with a characteristic admittance G with the requirement that this line match

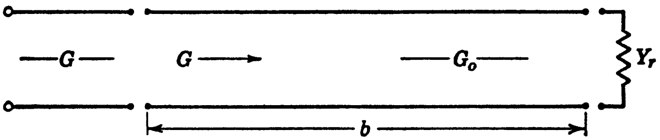


FIG. 11–23. Short section of line.

a receiving admittance Y_r. Reference to Fig. 11–22 shows that two points on the circle are thus known, G (at c) and Y_r (at a). Knowing the location of these two points, the circle may now be constructed. Draw a line from c to a, and construct the perpendicular bisector of this line. The point of intersection of this bisector with the G axis locates the center of the circle, point f. With f as a center and f–a as radius the circle is drawn. The characteristic admittance of the matching section G_0 is then found as the square root of the scalar distances od and oc. Using equations 11–83 and 11–84, the values of od and oc may be written as

$$od = \frac{G_r^2 + B_r^2 + G_0^2}{2G_r} + \sqrt{\left(\frac{G_r^2 + B_r^2 + G_0^2}{2G_r}\right)^2 - G_0^2}$$

$$oc = \frac{G_r^2 + B_r^2 + G_0^2}{2G_r} - \sqrt{\left(\frac{G_r^2 + B_r^2 + G_0^2}{2G_r}\right)^2 - G_0^2}$$

On multiplication of these two expressions it is found that $od \times oc = G_0^2$, or

$$G_0 = \sqrt{od \times oc} \tag{11-85}$$

Thus G_0 can be determined semigraphically from the given values of G and Y_r.

The value of b, which is the angular length of the matching section and proportional to the arc of the circle measured clockwise from Y_r, can be determined by a rearrangement of equation 11–79:

$$\tan b = \frac{G_0(G_r - G)}{-(GB_r + G_rB)}$$

or

$$b = \tan^{-1} \frac{G_0(G_r - G)}{-(GB_r + G_rB)} \tag{11-86}$$

In the event that Y_r is a pure conductance the problem reduces to that treated in Art. 92. A circle-diagram method similar to the above may be used when the characteristic admittance of the line to be matched is not a pure conductance.

Illustrative Example. Let it be required to find the value of G_0 for a matching section to match a line of characteristic admittance $G = 0.002$ mho ($Z = 500$ ohms resistance) to a terminating admittance of $Y_r = 0.006 - j0.002$ mho ($Z_r = 150 + j50$ ohms). The construction is shown in Fig. 11–24. First the points G and Y_r are located and connected by the line ca. The perpendicular bisector of this line is erected locating the center of the circle at point f. Then with f–a as radius and f as center a circle is drawn thus locating the point d. From the diagram $oc = 0.002$ and $od = 0.007$ whereupon

$$G_0 = \sqrt{0.007 \times 0.002} = 0.00374 \text{ mho}$$

which is equivalent to a characteristic impedance of 267.4 ohms.

The value of b is determined from equation 11–86. $G_r = 0.006$, $B_r = -0.002$, $G = 0.002$, $B = 0$, $G_0 = 0.00374$.

$$b = \tan^{-1} \frac{0.00374 \times (0.006 - 0.002)}{-(0.002 \times -0.002)}$$

$$= \tan^{-1} 3.74 = 75.02°$$

which is equivalent to 0.208λ.

As a further illustration, let the problem of Art. 93, wherein a resistance of $R_g = 200$ ohms is to be matched to a load resistance of $R_r = 700$ ohms, be worked using the graphical method.

$Z_g = R_g = 200$ ohms, from which Y $(= G + jB) = 0.005 + j0$ mho and $Z_r = R_r = 700$ ohms, from which Y_r $(= G_r + jB_r) = 0.00143 + j0$ mho.

From the data it is seen that both points, G and Y_r ($= G_r$), now lie on the diameter of the circle, and their magnitudes are those of oc and od, namely, 0.005 and 0.00143, giving

$$G_0 = \sqrt{0.00143 \times 0.005} = 0.002675 \text{ mho}$$

or

$$Z_0 = 374 \text{ ohms}$$

The length of the matching section is found from

$$b = \tan^{-1} \frac{0.002675 \times (0.00143 - 0.005)}{0} = 90°$$

which is a quarter-wavelength line.

The characteristic impedance and the length of the line each check the values found according to Art. 93.

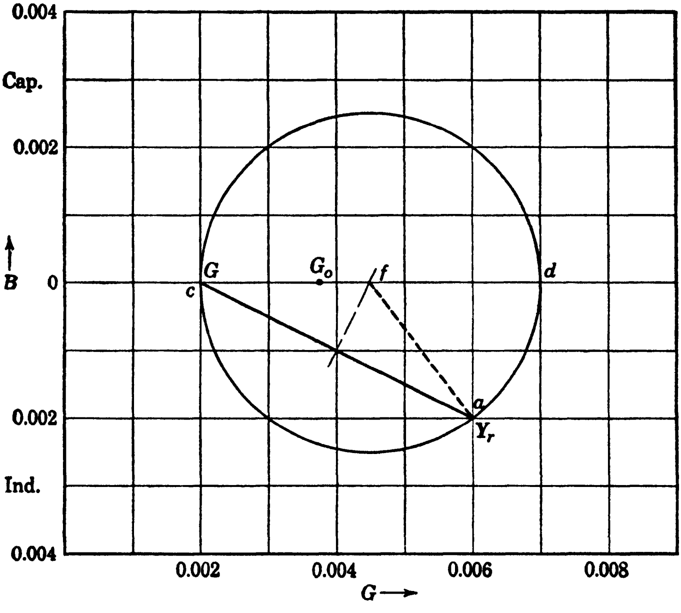


FIG. 11–24. Circle diagram for illustrative example of Art. 98.

99. The Matching Stub. The analysis of the matching stub given in Art. 96 may be complemented by applying the graphical method outlined in Art. 98.

Let the transmission line be represented as in Fig. 11–25. The circle diagram based on this line and its terminating admittance is shown in Fig. 11–26 for a resistive-capacitive termination. If the length b of the line is now increased until such a point as c on the circle is reached, it is seen that Y has a real component equal to OG_0 and

a reactive component equal to *ac*. If a susceptance of the opposite kind and of the same magnitude as *ac* is connected across the line at this point, the net admittance at *A* (of Fig. 11–25) becomes merely

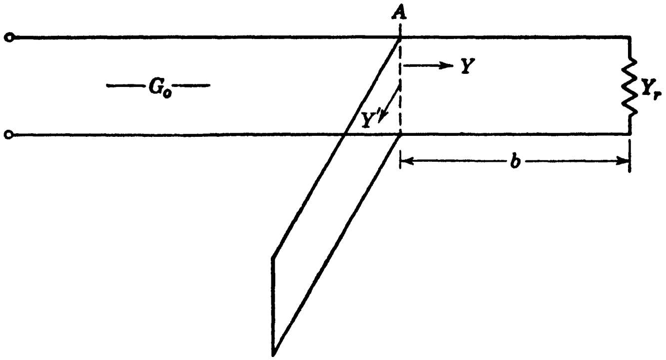


FIG. 11–25. Transmission line matching by means of short-circuited stub.

G_0, thus matching the line at that point. Since *ac* is here inductive, a capacitive susceptance is placed across the line. As presented in Art. 96, it is convenient to use a short section of short-circuited line for the

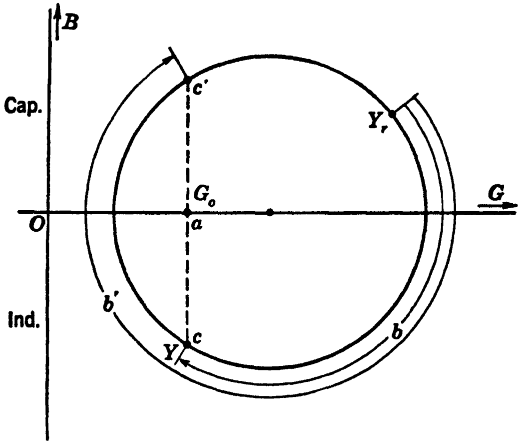


FIG. 11–26. Circle diagram for line of Fig. 11–25.

matching stub, and for the stub to be capacitive it will be between one-quarter and one-half wavelength.

Instead of stopping at point *c* and placing a capacitive susceptance equal to *ac* across the line at point *b* it would be possible to proceed to point *c′* where *Y* again has a real component equal to OG_0 and a reactive component equal to *ac′*, the reactive component now being

capacitive, however. Since ac' is capacitive, an inductive susceptance equal in magnitude to ac' is placed across the line at the distance b' from the load. The short-circuited stub, now being inductive, will be less than a quarter wavelength.

As an illustration, let the problem of Art. 97, in which a line of characteristic impedance $Z_{01} = 600$ ohms is to be matched to a load of $R = 200$ ohms, be worked by the graphical method. The point of

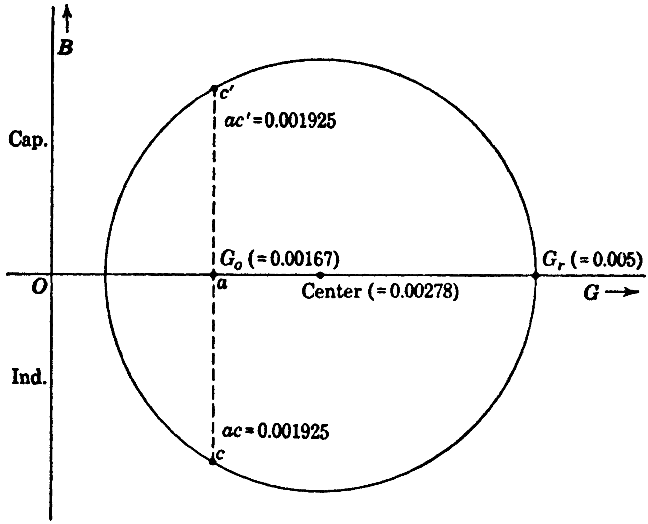


FIG. 11–27. For use in connection with illustrative example of Art. 99.

contact and the length of the stub are to be determined. The characteristic impedance of the stub is to be the same as that of the line.

$$Y_r = G_r + jB_r = 0.005 + j0$$

$$Y_0 = Y = G_0 + jB_0 = 0.00167 + j0$$

The center of the circle is found from equation 11–83 to be at coordinates 0, and

$$\frac{0.005^2 + 0 + 0.00167^2}{2 \times 0.005} = 0.00278$$

The circle is constructed as shown in Fig. 11–27, passing through point Y_r (= G_r). The distance ac (= $-B$) represents 0.001925 mho inductive susceptance which is likewise the value of capacitive susceptance required for matching.

The length of the stub may be obtained from equation 11–63, noting

that the characteristic impedance of the stub is to be the same as that of the line. The impedance of the stub is

$$\frac{1}{0.001925} \qquad 520 \text{ ohms capacitive reactance}$$

From equation 11–63,

$$\beta y_{s1} = \tan^{-1} \frac{X''}{Z_{02}} = \tan^{-1} \frac{-520}{600} = 139.1°$$

whereupon $y_{s1} = 0.386\lambda$.

The value of b is obtained from equation 11–86.

$$b = \tan^{-1} \frac{0.00167 \times (0.005 - 0.00167)}{-(0.005 \times -0.001925)} = 30°$$

giving $S_1 = 0.0833\lambda$.

By progressing to point c' of Fig. 11–27, the distance ac' $(= +B)$ represents 0.001925 mho capacitive susceptance, whereupon the matching stub must be made to be 520 ohms inductive reactance.

From equation 11–63,

$$\beta y_{s2} = \tan^{-1} \frac{520}{600} = 40.9°$$

giving $y_{s2} = 0.1135\lambda$.

From equation 11–86,

$$b = \tan^{-1} \frac{0.00167 \times (0.005 - 0.00167)}{-(0.005 \times 0.001925)} = 150°$$

and $S_2 = 0.417\lambda$.

These two possible stubs with their respective points of attachment to the line check the values obtained in Art. 97.

100. Matching by Means of Two Stubs. The method of matching which has been treated in Arts. 96 and 99 requires that it be possible to adjust the position of the stub along the line. However, when either a coaxial line or a wave guide is to be matched to a load, it may not be possible or convenient to adjust the position of the stub. With these lines a method of matching which uses two stubs at fixed positions near the end of the line may often be employed. This double-stub method can be explained most readily by means of the circle diagram.

Let a line of characteristic admittance G_0 which is terminated in a conductance G_r be represented as in Fig. 11–28. Let its admittance circle be the circle A. The object is to place a stub at some point on the line, as at C, such that there will be presented to the line to the

left of C an admittance G_0, thus producing a condition of matched impedances. At the end of the line, across the terminating admittance, a tuning stub is attached which can be adjusted to represent inductive

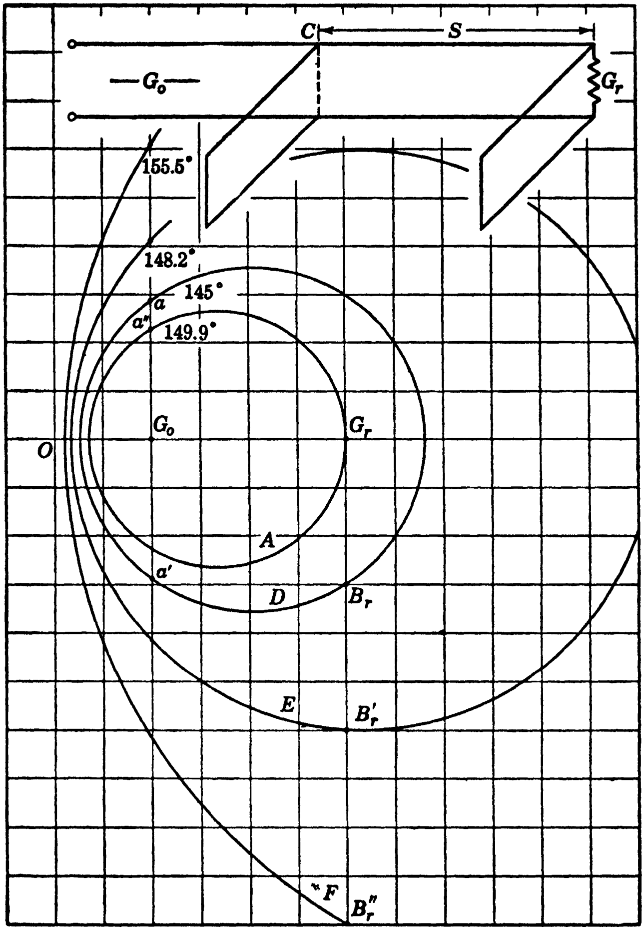


FIG. 11-28. Transmission line matching by means of two stubs.

susceptance B_r across G_r. The termination now becomes $G_r - jB_r$, and the new admittance circle is D.

The centers of circles A and D and their radii may be found by use of equations 11-83 and 11-84. Circle A must of course pass through G_r and circle D must go through $G_r - jB_r$.

Proceeding back along the line, the admittance will be given by points on the circle D, and, in particular, one of these points is a on the

line drawn vertically above G_0. If an inductive susceptance equal to G_0–a is placed across the equivalent point on the line, the admittance of the line at point C measured toward the load will be G_0, which was desired. Instead of proceeding back to point a of the circle and placing an inductive susceptance equal to G_0–a across the line at that point a capacitive susceptance could be placed corresponding to point a', which is vertically below G_0. However, by using the greater length as at a a stub less than a quarter wavelength may be employed, whereas the length represented by the point a' would require a stub of one-quarter to one-half wavelength.

Practically, the distance S is fixed by the construction of the line so that the physical separation of the two stubs cannot be changed. However, the effective distance can be adjusted by the use of the tuning stub at the end of the line. The procedure may be best shown by means of an illustrative example. Assume in the illustration of Fig. 11–28 that $G_0 = 0.002$ mho, $G_r = 0.006$ mho, and $B_r = 0$. The center of the circle is found from equation 11–83 to be at coordinates 0, and

$$\frac{0.006^2 + 0 + 0.002^2}{2 \times 0.006} = 0.00333$$

With this point as center the circle A is drawn passing through the point G_r. (See Fig. 11–28.) The distance G_0–a'' ($= B$), as measured, represents 0.0023 mho capacitive susceptance. This is the value of the inductive susceptance required for matching. The value of b is obtained from equation 11–86.

$$b = \tan^{-1} \frac{0.002(0.006 - 0.002)}{-(0.006 \times 0.0023)} = 149.9°$$

or $S = 0.417\lambda$.

Now assume that an inductive susceptance of $B_r = -0.003$ mho is placed across G_r. From equation 11–83 the center of the circle is found to be at coordinate 0, and

$$\frac{0.006^2 + 0.003^2 + 0.002^2}{2 \times 0.006} = 0.00408$$

Circle D is drawn passing through point $G_r - jB_r$. The distance G_0–a ($= B$) represents 0.0029 mho capacitive susceptance. The value of b is

$$\begin{aligned} b &= \tan^{-1} \frac{0.002(0.006 - 0.002)}{-(0.002 \times -0.003 + 0.006 \times 0.0029)} \\ &= 145° \end{aligned}$$

or $S = 0.403\lambda$. Any negative value of B_r greater than 0.003 mho, such as B_r' ($= -0.006$ mho) gives a value of b greater than 145°. Thus it is seen that the effective length S can be changed by variation of the stub at the end of the line. In the illustrative example the minimum effective length of S is given by $b = 145°$ or 0.403λ; that is, the stubs cannot be placed closer together than 0.403λ.

Thus in this example, if the stubs are placed so that the distance S is somewhat greater than 0.403λ, then matching will be possible. However, if they are placed at points slightly less than 0.403λ, then no adjustment of the stubs could produce matching.

101. The Transmission-Line Calculator.[4] A very useful tool in transmission-line calculations is the circular slide-rule chart based on the circle-diagram method of solving transmission-line problems. The chart which is the basis of this calculator depends on a transformation of the ordinary R–X coordinate plane, used for the expression of complex impedances, into a new curvilinear system through a conformal transformation. The present treatment will cover impedance transformations in terms of the length of the line and matching by means of a single stub.

The input impedance of a high-frequency lossless line in terms of its terminating impedance and line length is given by equation 6–46,

$$Z_s = Z_0 \frac{Z_r \cos b + jZ_0 \sin b}{Z_0 \cos b + jZ_r \sin b}$$

Since Z_0 is a constant for any particular problem, impedances will be written as multiples of Z_0 rather than in the usual direct form. Thus the input impedance will be written

$$Z = \frac{Z_s}{Z_0} = \frac{Z_r \cos b + jZ_0 \sin b}{Z_0 \cos b + jZ_r \sin b} \qquad [11\text{–}87]$$

where $b = \beta S$.

When written in terms of exponentials and rearranged, equation 11–87 becomes

$$Z = \frac{1 + \dfrac{Z_r - Z_0}{Z_r + Z_0}\epsilon^{-j2b}}{1 - \dfrac{Z_r - Z_0}{Z_r + Z_0}\epsilon^{-j2b}} = \frac{1 + K\epsilon^{-j2b}}{1 - K\epsilon^{-j2b}} \qquad [11\text{–}88]$$

$$= \frac{1 + W}{1 - W} \qquad [11\text{–}89]$$

where $W = K\epsilon^{-j2b}$ [11–90]

[4] "Transmission Line Calculator," by P. H. Smith, *Electronics*, Jan. 1939. "An Improved Transmission Line Calculator," by P. H. Smith, *Electronics*, Jan. 1944.

and

$$K = \frac{Z_r - Z_0}{Z_r + Z_0} = K \underline{/\psi} \qquad [11\text{–}43]$$

In terms of $Z' = \dfrac{Z_r}{Z_0}$

$$K = \frac{\dfrac{Z_r}{Z_0} - 1}{\dfrac{Z_r}{Z_0} + 1} = \frac{Z' - 1}{Z' + 1} \qquad [11\text{–}91]$$

(Note also that equation 11–88 may be obtained as the quotient of equation 11–41 and equation 11–42.) In the above equations, K is in

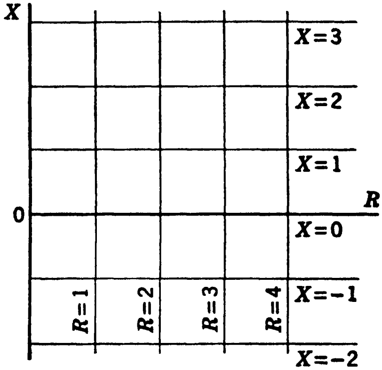


FIG. 11–29. R–X field in Cartesian coordinate system.

general a complex quantity of the form $K\epsilon^{j\psi}$ expressible on the W plane as a vector, and ϵ^{-j2b} is a term which produces rotation of K in accordance with the value of b, the angular length of the line under consideration. Equation 11–89 can be solved for W also, giving

$$W = \frac{Z - 1}{Z + 1} \qquad [11\text{–}92]$$

Equations 11–89 and 11–92 represent a linear transformation between the Z- and W-planes. That is, $W = U + jV$ can be plotted on the W plane where the coordinate axes are U and V, or Z can be plotted on the Z plane where the coordinates are the familiar R and X.

In Fig. 11–29 is represented the usual R–X coordinate system where the lines, R = constant and X = constant, are of course perpendicular. It is of interest now to determine the shape of the corresponding W

curves, of Fig. 11–30, as given by equation 11–92. These curves will again be characterized by setting R = constant and X = constant. For this purpose equation 11–92 is written

$$W = U + jV = \frac{Z - 1}{Z + 1} = \frac{R + jX - 1}{R + jX + 1} \qquad [11\text{–}92a]$$

or $$U + UR - VX + j(RV + UX + V) = R + jX - 1$$

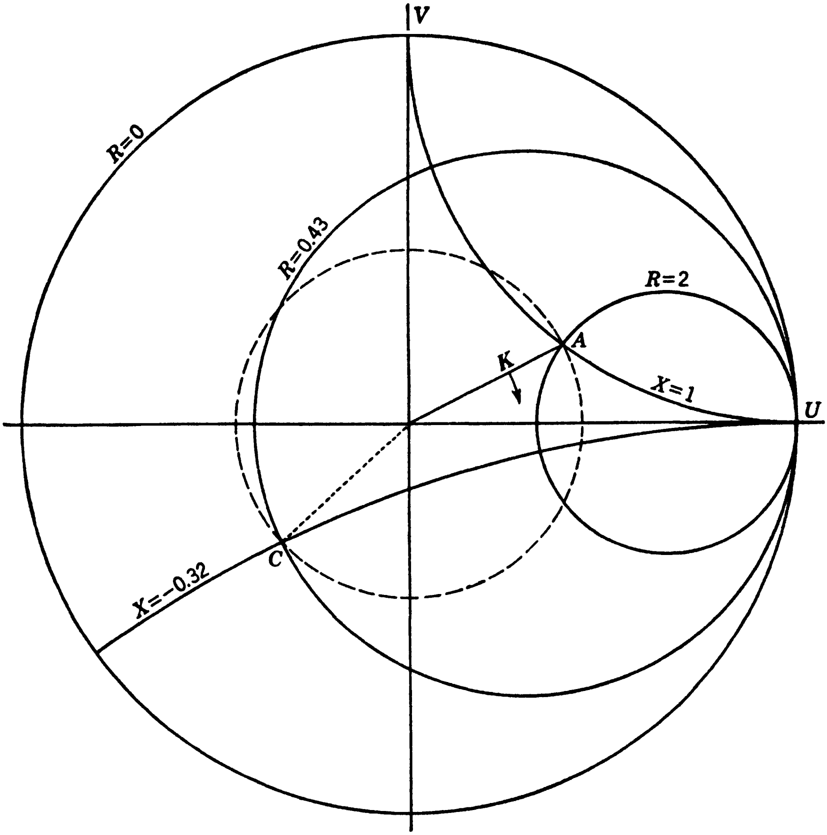


FIG. 11–30. Transmission-line chart for determining input impedance of high-frequency line. Example of Art. 101.

By separating real and quadrature terms, two equations are obtained:

$$U + UR - VX = R - 1 \qquad [11\text{–}93]$$

$$RV + UX + V = X \qquad [11\text{–}94]$$

Curves for constant R are determined by eliminating X from equa-

tions 11–93 and 11–94. Then

$$X = \frac{U + UR - R + 1}{V}$$

and

$$X = \frac{RV + V}{1 - U}$$

or, setting the right-hand members equal,

$$U^2(R + 1) - U(2R) + (R - 1) = -V^2(R + 1)$$

By completing the square in U and rearranging, there is obtained

$$\left[U - \frac{R}{R + 1}\right]^2 + V^2 = \frac{1}{(R + 1)^2} \qquad [11\text{–}95]$$

Equation 11–95 is the equation of a family of circles, the centers of which are at

$$\left.\begin{aligned} U &= \frac{R}{R + 1} \\ V &= 0 \end{aligned}\right\} \qquad [11\text{–}96]$$

and whose radii are

$$r = \frac{1}{R + 1} \qquad [11\text{–}97]$$

These circles all pass through the point 1,0, and two have been drawn in Fig. 11–30 for $R = 2$ and $R = 0.43$ ohms.

The curves for constant X are determined in a similar manner, by eliminating R from equations 11–93 and 11–94. The following is obtained, after completing the squares and rearranging,

$$(U - 1)^2 + \left(V - \frac{1}{X}\right)^2 = \frac{1}{X^2} \qquad [11\text{–}98]$$

Equation 11–98 is the equation of a family of circles whose centers are at

$$\left.\begin{aligned} U &= 1 \\ V &= \frac{1}{X} \end{aligned}\right\} \qquad [11\text{–}99]$$

and whose radii are

$$r = \frac{1}{V} \qquad [11\text{–}100]$$

where X may be positive or negative. It is seen that these circles all pass through the point 1,0 also. Two are shown in Fig. 11–30, for $X = 1$ and $X = -0.32$ ohm.

The field of Fig. 11–29 represents the usual R–X system in which every point to the right of the X axis corresponds to a possible value of $\boldsymbol{Z}\ (= R + jX)$. The field of Fig. 11–30 represents an R–X system, also, now distorted and made up of circles, in which every point within the unit circle, $R = 0$, corresponds not only to a particular value of $\boldsymbol{Z}$ but also to the corresponding value of $\boldsymbol{W}\ (= U + jV)$. Thus the field of Fig. 11–30 is a *translation field* for transforming back and forth from one system to another. This transformation is known as a *conformal transformation* and is such as to leave angles unchanged. For instance at the point A in both systems is represented the same value of $\boldsymbol{Z}$, and the lines of constant R and X are perpendicular.

The point A, as stated, represents two quantities. It represents $\boldsymbol{Z} = 2 + j1$, and it also represents a certain value of $\boldsymbol{W} = U + jV$. This value of $\boldsymbol{W}$ can be expressed by equation 11–90 in terms of $\boldsymbol{Z}_r$, $\boldsymbol{Z}_0$ and b, or by equation 11–92 in terms of $\boldsymbol{Z} = R + jX$. Equation 11–90 states that the locus of $\boldsymbol{W}$, expressed in terms of $\boldsymbol{Z}_r$, $\boldsymbol{Z}_0$ and b, is a circle about the origin. When the terminating impedance is held constant, but the electrical line length varied, the values of U and V will lie on a circle with the origin as center. The value of $\boldsymbol{K}$ can change only if the terminating impedance changes, and ϵ^{-j2b} can change only if the line length changes. It should also be noted that one complete revolution is made when b varies over 180° or the electrical length of the line is varied by a length equal to $\lambda/2$. The circle for $\boldsymbol{Z}_r = 140 + j70$ ohms and $\boldsymbol{Z}_0 = 70$ ohms is drawn (dashed) in Fig. 11–30.

Thus there is now set up a graphical construction which has certain advantages since $\boldsymbol{W}$ appears as a circle with a uniform angle scale. Distance from the load may be measured by merely rotating in a clockwise direction (as indicated by the negative exponent in equation 11–90) from any particular starting point. It is possible now to fix in this field a certain value of load $\boldsymbol{Z}$ corresponding to $b = 0$, such as at A. Going back toward the generator is represented by following the dashed circle in a clockwise direction for the required distance, say to the point C where the value of $\boldsymbol{Z}$, the input impedance at that point, may be read on the R–X coordinate system.

A numerical example will be given to clarify the discussion. Let it be required to find the input impedance to a short high-frequency line in which $\boldsymbol{Z}_r = 140 + j70$ ohms, $\boldsymbol{Z}_0 = 70 + j0$ ohms, and $b = 82°$. The terminal or load impedance in terms of $\boldsymbol{Z}_0$ is

$$\boldsymbol{Z}' = 2 + j1$$

This value is set at point A on Fig. 11–30 at the intersection of $R = 2$ and $X = +1$ circles. From equations 11–90 and 11–91,

$$
\begin{aligned}
W &= K\epsilon^{-j2b} \\
&= \frac{Z' - 1}{Z' + 1}\epsilon^{-j2b} \\
&= \frac{2 + j1 - 1}{2 + j1 + 1}\epsilon^{-j2(82°)} \\
&= 0.447 \underline{/26.57°} \underline{/-164°} \\
&= K \underline{/-164°}
\end{aligned}
$$

It is seen then that, in terms of U and V, K is given by the point A and that W is obtained by rotating K clockwise from this point through 164 mechanical degrees (degrees on the chart) corresponding to 82° on the line, locating point C. Here

$$W = 0.447 \underline{/-137.43°}$$

and at this point C

$$Z = 0.43 - j0.32$$

and

$$Z_s = (0.43 - j0.32) \times 70 = 30.1 - j22.4 \text{ ohms.}$$

This value of Z_s should be checked by means of equation 6–46.

It is evident then, that the use of the chart for determining impedances at any point involves a process of transferring back and forth between the two coordinate systems which are superposed so that every pair of W values corresponds to an appropriate pair of Z values. First, one sets the value of Z on the W plane by using the R and X values; then the appropriate rotation is added, corresponding to the length of line, which rotates the radius vector on a uniform scale to a new value of W which is read off *in terms of* Z.

The process will work in either direction. For instance, an input impedance may be known at some distance b ahead of the load. This value of Z is set on the chart, and a rotation of b is made in the *counter-clockwise* direction leading one to the value of the load impedance Z'.

Determination of an Unknown Impedance from the Standing Wave Ratio. At that point on a line where the standing wave (voltage) is a maximum the per unit resistance has the value of the standing wave ratio. This fact may be used to determine an unknown impedance, such as Z_r, provided the characteristic impedance of the line and the distance to the first voltage maximum are known.

At V_{max}, $2\beta x_{max} = \psi$

and, from equation 11–44,

$$V_{max} = I_r \cdot \frac{Z_r + Z_0}{2} \epsilon^{j\beta x}[1 + K]$$

and, from equation 11–45,

$$I_{min} = I_r \cdot \frac{Z_r + Z_0}{2Z_0} \epsilon^{j\beta x}[1 - K]$$

$$\frac{V_{max}}{I_{min}} = Z = Z_0 \cdot \frac{1 + K}{1 - K}$$

and

$$\frac{Z}{Z_0} = \frac{1 + K}{1 - K} = N \quad \text{(a scalar quantity)}$$

Hence at the point of voltage maximum the per unit impedance is resistive and has the value of the standing wave ratio N.

In the previous illustrative example it is seen that $K\underline{/\psi} = 0.447\underline{/26.57°}$ and $N = 2.62$. Refer to Fig. 11–30. Upon rotating $\boldsymbol{K}$ clockwise through 26.57°, thereby locating the position of the standing wave (voltage) maximum, the value of the per unit $\boldsymbol{Z}$ is found to be $2.62 + j0$ and the per unit resistance has the value of the standing wave ratio.

In Fig. 11–31 is shown the Smith chart in which circles of constant R and X are solid and circles of constant N are dashed. The values of N may be read along the horizontal axis between the points 1.0 and ∞. Note that around the outside of the chart is a scale marked off in terms of decimal parts of a wavelength for convenience in making calculations. With its use a protractor is unnecessary.

As an illustrative example, let it be required to determine the value of an unknown load impedance on a line which has a characteristic impedance of $300\ \underline{/0°}$ ohms and a standing wave ratio of 4.5. The distance from the load to the first voltage maximum is 104° or 0.289λ. Refer to Fig. 11–31. The per unit impedance at the point of the standing wave maximum is given by point A where the standing wave ratio is 4.5. On moving around the circle having the standing wave ratio of 4.5 a distance of 0.289λ *toward the load*, the per unit impedance becomes that specified by point B ($= 0.236 - j0.236$ ohms). Thus $\boldsymbol{Z}_r$ is $100\ \underline{/-45°}$ ohms.

Note. In Fig. 11–31, the per unit impedance at the point of the standing wave minimum is given by point A' ($= 1/N$) and of course the distance from the load to the first voltage minimum is 0.039λ. The same per unit im-

pedance is then obtained by moving around the 4.5-standing-wave-ratio circle a distance of 0.039λ *toward the load.*

The charts of Figs. 11–31 and 11–32 are available for working problems given at the end of the chapter.

The Matching Stub. For the solution of the matching-stub problem it will again be convenient to work with admittances. The input admittance of a line in terms of the characteristic admittance Y_0 is, from equation 11–88,

$$Y = \frac{Y_s}{Y_0} = \frac{1 - K\epsilon^{-j2b}}{1 + K\epsilon^{-j2b}} \qquad [11\text{–}101]$$

$$= \frac{1 - W}{1 + W} \qquad [11\text{–}102]$$

where $W = K\epsilon^{-j2b}$

Or, from equation 11–102,

$$W = \frac{1 - Y}{1 + Y} \qquad [11\text{–}103]$$

Then

$$W = U + jV = \frac{1 - Y}{1 + Y} = \frac{1 - G - jB}{1 + G + jB} \qquad [11\text{–}103a]$$

By writing $Y = G + jB$ and solving equation 11–103a in a manner similar to that employed in the solution of equation 11–92a it may be shown that the chart of Fig. 11–30 is applicable to the use of admittances by simply substituting G's for R's and B's for X's, $+B$ for $+X$, and $-B$ for $-X$.

In order to illustrate the procedure involved let it be proposed to design a short-circuited matching stub and to determine the point of contact in order to match a line whose characteristic impedance is $Z_0 = 600\underline{/0°}$ ohms to a load of $Z_r = 150 + j150$ ohms. The characteristic impedance of the stub is to be the same as that of the line. (See Art. 97.)

$Z_0 = 600 + j0$ ohms $\qquad Y_0 = 0.001667$ mho

$Z_r = 150 + j150$ ohms $\qquad Y_r = 0.00333 - j0.00333$ mho

$$\frac{Y_r}{Y_0} = 2 - j2$$

(*a*) Locate $Y_r/Y_0 = 2 - j2$ as shown at A on Fig. 11–33. Rotate K clockwise until it intersects the $G = 1$ circle at C $(= 1 - j1.58)$. The stub is placed on the transmission line at this point since at the point of attachment

G must equal $1/Z_0$ (see equation 11-52) or $G = Y_0$ and in terms of the characteristic admittance $G = 1$. The angle passed through in going from A to C represents a length of line corresponding to $2b$. In this instance the angle is $2b = 22°$ or $b(= \beta S_{\text{stub 1}}) = 11°$.

(b) Or rotate K until it again intersects the $G = 1$ circle at point D ($= 1 + j1.58$). The angle passed over is $2b = 278.8°$ or $b(= \beta S_{\text{stub 2}}) = 139.4°$.

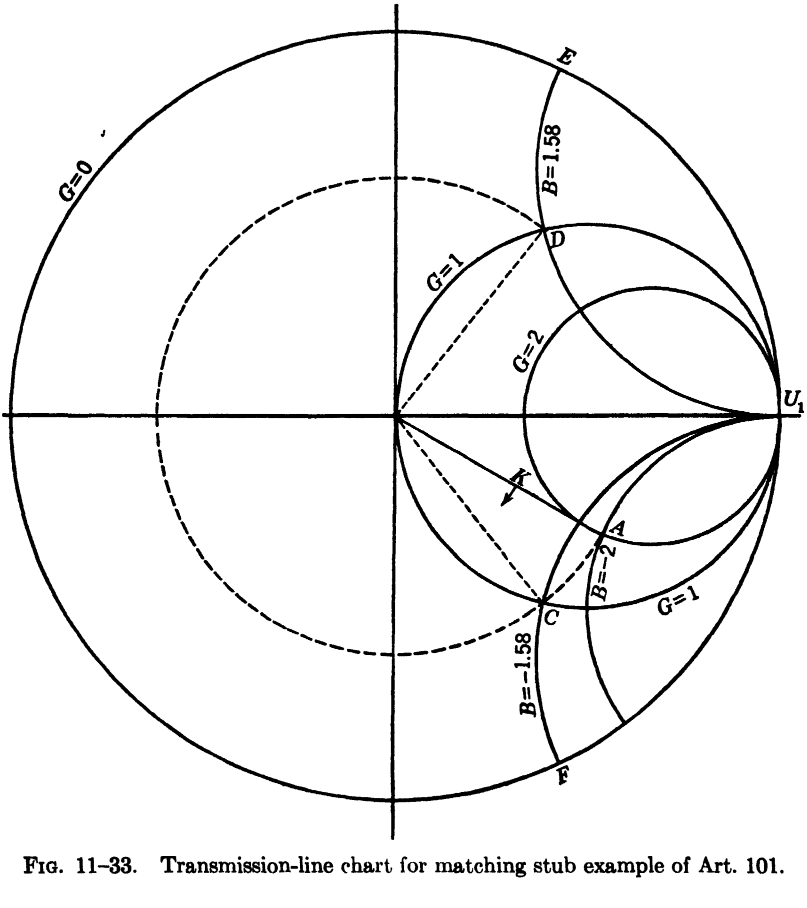


FIG. 11-33. Transmission-line chart for matching stub example of Art. 101.

(c) To determine the length of short-circuited stub corresponding to placement (a), start at point U_1 (where $Y_{r\text{stub}}/Y_0 = \infty$), and proceed clockwise around the $G = 0$ circle until the intersection with $B = +j1.58$ is reached at E. (Reactance of the type opposite to that of the line is required.) The angle passed through represents a length of stub corresponding to $2b_{\text{stub 1}}$. In this case the angle is $2b_{\text{stub 1}} = 295°$ or $b_{\text{stub 1}} = 147.5°$.

(d) To determine the length of stub corresponding to placement (b), start at point U_1, and proceed clockwise around the $G = 0$ circle until the inter-

section with $B = -j1.58$ is reached at F. The angle passed through represents a length of stub corresponding to $2b_{\text{stub 2}}$. The angle is $2b_{\text{stub 2}} = 64.5°$ or $b_{\text{stub 2}} = 32.3°$.

These values agree with the analytical results obtained in Art. 97.

PROBLEMS

11-1. Design an H pad that will match a load resistance of 100 ohms and have an attenuation of 4 decibels.

11-2. A generator having an internal resistance of 500 ohms is to be matched to a load impedance of 200 ohms resistance. Design a two-element (L-type) reactive matching network to match this load to the generator.

11-3. Design an L section which will produce a reflection match between a generator of impedance $Z_g = 60 + j30$ ohms and a load impedance of $Z_r = 600 - j300$ ohms. What decibel gain or loss results from inserting the L section?

11-4. Design an L section which will produce a reflection match between a generator of impedance $Z_g = 600 - j300$ ohms and a load impedance of $Z_r = 60 + j30$ ohms. What decibel gain or loss results from inserting the L section?

11-5. (*a*) Design an L section which will produce a reflection match between a generator of impedance $Z_g = 200 \underline{/0°}$ ohms and a load impedance of $Z_r = 67 \underline{/-26.6°}$ ohms. What change in decibels occurs from inserting the matching section?

(*b*) Design a reactive L section which will effect a conjugate match. What change in decibels now occurs from inserting the matching section?

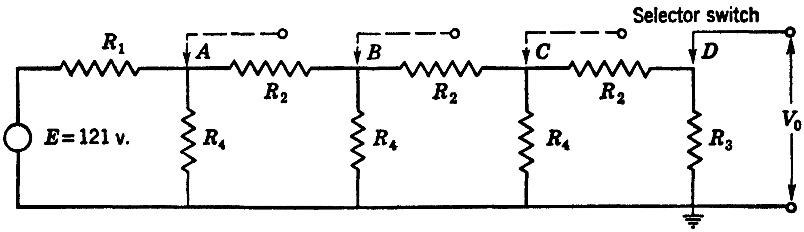


FIG. 11-34. For use in connection with Prob. 11-8.

11-6. (*a*) Design an L section which will produce a reflection match between a generator of impedance $Z_g = 67 \underline{/-26.6°}$ ohms and a load impedance of $Z_r = 200 \underline{/0°}$ ohms. What change in decibels occurs from inserting the matching section?

(*b*) Design a reactive L section which will effect a conjugate match. What change in decibels now occurs from inserting the matching section?

11-7. A transmission line whose characteristic impedance is 600 ohms resistance is to be connected to another line whose Z_0 is 200 ohms resistance.

(*a*) Design an L-type resistance matching network.

(*b*) Design an L-type reactive matching network.

(*c*) What gain or loss in decibels occurs due to the insertion of each type of matching network?

11-8. Design a voltage attenuator as shown in Fig. 11-34 to meet the following requirements:

1. $V_B = 0.1\ V_A$, $V_C = 0.1\ V_B$, and $V_D = 0.1\ V_C$.

2. The resistance looking back into the network from the V_0 terminals shall be the same whether the selector switch is set at either the A, B, C, or D positions.

(a) Determine three linear relationships between the resistors, R_1, R_2, R_3, and R_4 from which the general requirements can be met.

(b) Determine the resistance looking back into the network from the V_0 terminals if $R_1 = 110$, $R_2 = 99$, $R_3 = 11$, and $R_4 = 110/9$ ohms with the selector switch set at positions A, B, C, and D.

11–9. A generator has a resistance of 600 ohms and is to supply power to a load resistor of 100 ohms.

(a) Design a reactive L section to match these resistors.

(b) Design a reactive T section to effect the match.

(c) What gain in decibels occurs due to the insertion of each type of matching network?

11–10. Specify the elements of a reactive T section which will match a generator of $Z_g = 500\,\underline{/0°}$ ohms to a load $Z_r = 70\underline{/0°}$ ohms and produce a phase shift of 137°. Z_1 is to be set at 800 ohms, and the frequency is 10 megacycles per second.

11–11. The two resistances in Prob. 11–9 are to be matched at 20 megacycles per second by using a quarter-wavelength line made of ⅛-inch-diameter wire. What are the length and spacing of the wires?

11–12. An impedance of $100 + j250$ ohms is to be converted to its conjugate by the use of a quarter-wavelength line. Can this be done, and if so what are the characteristics of the line?

11–13. Find the value of K for a line having a characteristic impedance of $300\,\underline{/0°}$ ohms which is terminated in an impedance of $100\,\underline{/-45°}$ ohms. What is the voltage ratio, V_{max}/V_{min}, for this line and its terminating impedance?

11–14. An antenna feeder is 1000 feet long and is made up of two parallel wires ⅛ inch in diameter and spaced 6 inches center to center. It supplies power at 5 megacycles per second to an antenna whose effective resistance is 100 ohms. Design a short-circuited stub for impedance matching, and specify at what point it should be attached to the line. Assume both lines have the same $Z_0 (= \sqrt{L/C})$.

11–15. The antenna feeder in Prob. 11–14 is to be matched to the 100-ohm resistance load at a frequency of 10 megacycles per second. The stub line has a Z_0 of 150 ohms resistance. Find the length of the stub and the point at which it must be connected to the line.

11–16. A line having a characteristic impedance of 475 ohms resistance is terminated in an impedance of $100 + j100$ ohms. This line is to be matched to its terminating impedance by the use of a short-circuited stub which has the same characteristic impedance as the line. Determine the length of the matching stub and the distance from the receiver end of the line to the stub for each of the two correct stubbing points closest to the receiver by:

(a) the analytical method of Art. 96,

(b) the graphical method of Art. 99.

11–17. Experimental data obtained on a certain high-frequency line shows that $V_{max}/V_{min} = 4.26$ and the distance to the first voltage maximum is 0.208λ. Determine the location of short-circuited matching stubs, and give the lengths of these stubs.

11-18. A line whose characteristic impedance is $300 \angle 0°$ ohms is operating at 200 megacycles, and it is desired to use this line to measure an unknown impedance which is connected across the receiver terminals, at $x = 0$. Experimental data yield $V_{max} = 100$ units and $V_{min} = 20$ units, and the first voltage maximum (from $x = 0$) is located at 12.5 centimeters from the receiver terminals. Find the value of Z_r specifying both the magnitude and the phase angle of Z_r.

11-19. Determine the value of characteristic impedance necessary for a matching section which is to match a line of characteristic impedance $Z_0' = 475$ ohms resistance to a terminating impedance of $100 - j100$ ohms. Use the method of Art. 98. Specify, in terms of wavelength, the length of the section.

11-20. Solve Prob. 11-14 by the graphical method of Art. 99.

11-21. A line having a characteristic impedance of $300 \angle 0°$ ohms is terminated in an impedance of $100 \angle -45°$ ohms. This line is to be matched to its terminating impedance by the use of a short-circuited stub which has the same characteristic impedance as the line. Determine the length of the matching stub and the distance from the receiver end of the line to the stub for each of the two correct stubbing points closest to the receiver by:

(a) the analytical method of Art. 96,
(b) the plotted form as illustrated in Fig. 11-18,
(c) the graphical method of Art. 99.

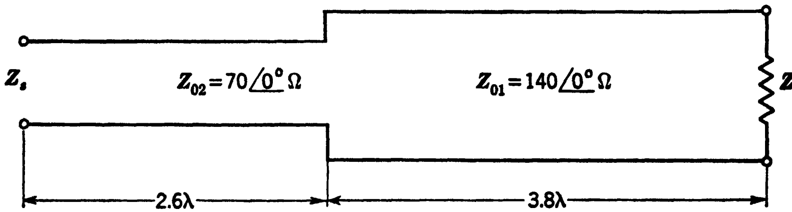


FIG. 11-35. For use in connection with Prob. 11-27.

11-22. The line and load impedance of Prob. 11-16 are to be matched by means of two stubs, one at the terminating impedance and the other $\frac{4}{10}$ of a wavelength from the terminating end of the line. Find the length of each stub necessary for matching. Each stub is to have the same characteristic impedance as that of the line.

11-23. Using the Smith chart, find the input impedance of a line for which $Z_0 = 300 + j100$ ohms, $\beta S = 338°$, $\alpha = 0$, and whose termination, Z_r, is $100 - j200$ ohms.

11-24. A line is $\frac{5}{8}$ wavelength long and must have an input impedance of $600 + j300$ ohms. $Z_0 = 300 \angle 0°$ ohms and $\alpha = 0$. Find the terminating impedance using the Smith chart.

11-25. Solve Prob. 11-14 by means of the Smith chart.

11-26. Solve Prob. 11-16 by means of the Smith chart.

11-27. A lossless line, 3.8λ long, which has a characteristic impedance of $Z_{01} = 140 \angle 0°$ ohms is terminated by an impedance Z_r and is connected at its other end to a lossless line 2.6λ long having a characteristic impedance of $Z_{02} = 70 \angle 0°$ ohms, as shown in Fig. 11-35.

(*a*) What must the value of Z_r be so that no standing waves are present on the 140-ohm line? What is the standing wave ratio on the 70-ohm line? What is the value of the input impedance Z_s?

(*b*) What must the value of Z_r be so that no standing waves are present on the 70-ohm line? What is the standing wave ratio on the 140-ohm line? What is the value of the input impedance Z_s?

11-28 A line which has a characteristic impedance of $500 \underline{/0°}$ ohms is terminated in an impedance of $500 + j500$ ohms. The line has an attenuation of 1 decibel per wavelength. What is the input impedance and standing wave ratio for a line which is 3 wavelengths long?

CHAPTER XII

ULTRAHIGH-FREQUENCY TRANSMISSION IN WAVE GUIDES — GENERAL

Of increasing importance in recent years has been the transmission of extremely high frequencies, usually referred to as UHF transmission. For this purpose two rather different systems have been used. One is the coaxial or concentric cable conductor which consists essentially of a tube with a rod placed inside of and concentric with it. These two elements constitute the two conductors of the line, and ordinary transmission theory can be applied, as shown in previous chapters. The second system makes use of a tube or pipe, rectangular, circular, or elliptical in cross section, with no internal conductor. The ordinary theory of transmission cannot be applied in general to this latter system. However, both can be handled on the electromagnetic-wave basis, using Maxwell's equations.[1]

A number of papers and books in recent years have treated the theory of such high-frequency transmission rather fully, and reference should be made to them for more complete information.

The second method mentioned, that of transmission through a hollow tube, is generally referred to as wave-guide transmission of ultrahigh frequencies. In a sense this is misnamed, as any method which confines the propagation to a certain direction may be called a wave-guide method. For instance, the ordinary parallel-wire system can be treated on the basis of electromagnetic waves guided by wires. However, the name has come to be associated definitely with this particular method and always means the propagation of electromagnetic waves through hollow tubes.

The material of this and the following chapters treats, in an elementary way, the theory of transmission through rectangular and cylindrical guides. Maxwell's equations will be needed for both types of transmission, but for the cylindrical type Bessel functions[2] will also be required.

102. Elements of Field Configuration. In beginning a discussion of the elementary theory of the so-called wave guide it is first advan-

[1] See Appendix VI.
[2] See Appendix VII.

tageous to emphasize that the propagation which takes place is through the dielectric material inside the guide. In considering the propagation of radio waves in space it is seen that whatever energy is transferred is conveyed by the advance of the electromagnetic wave. In radio this propagation is more or less in all directions unless a directive antenna is employed, whereupon the field is constrained to advance only in a relatively limited space. The wave guide serves merely to limit this field to still more definite bounds. The theory remains that of the propagation of the electromagnetic field which is based on Maxwell's equations.

In order to understand the material that follows it is essential to have in mind a relatively clear physical picture of the type of transmission to be considered. There are many modes of operation (or excitation) possible, and those treated in the following pages are selected as illustrative of the general problem. The ones selected are also the ones which are normally used in practice.

First, some elementary properties of electromagnetic fields must be reviewed. A fact which is fundamental to the general discussion here is that in the electromagnetic field the electric lines of force (the E vectors[3]) always cross the magnetic lines at an angle of 90°. This can be seen in an elementary manner by reference to equation A-63. Note from this equation that if the electric field is assumed to be entirely in the y direction, that is, $E_x = E_z = 0$, then the magnetic component H_y is identically zero, showing that H and E must be perpendicular. Thus, in a wave guide, the electric and magnetic lines are mutually perpendicular, regardless of the shape of the guide, and this fact in itself presents a means of obtaining an idea of the configuration of the field. The detailed configuration for a particular case however will depend on both the method of excitation used to establish the electromagnetic field and the boundary conditions existing at the walls of the guide.

103. Boundary Conditions and Field Distribution. Those conditions or restrictions which are imposed on an electromagnetic field at its boundaries are called boundary conditions. In an electromagnetic field which exists within an enclosure made of material having infinite conductivity the electric vectors (E) must always intersect the boundary of the enclosure at 90° angles. In other words, at the surface of the boundary there can be no tangential component of the electric field,

[3] In electromagnetic theory it is customary and convenient to employ the symbol E to represent the electric intensity (the negative of the potential gradient), even though the same symbol is used to represent emf (or potential difference) in circuit theory. The context of the subject matter will clearly indicate which of the two physical quantities is intended. See footnote (2), Chap. XIII, on notation.

since the material, having infinite conductivity, cannot support a potential gradient. This then constitutes a boundary condition. Also at high frequencies the current is found to flow in a very thin layer on the inside surface of the guide. (Refer to Fig. A–11, Appendix VI.) In this connection it is of interest to note that the conducting surfaces of wave guides are sometimes silver-plated to improve the conductivity and that only a very thin layer (in the order of 10^{-8} centimeter) is required to make a guide behave essentially as would a structure of solid silver.

Interest in the present treatment lies primarily in propagation through two types of wave guides, the rectangular and the cylindrical. In the

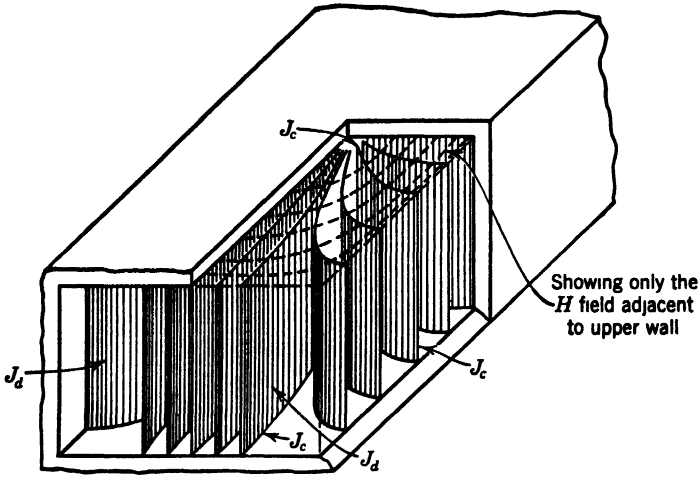


FIG. 12–1 Illustration of displacement current-density vectors (J_d) and conduction current-density vectors (J_c) in rectangular $TE_{0\,1}$ mode

rectangular wave guide which is considered first the simplest mode of operation is characterized by the fact that the electric lines and the associated displacement current-density vectors $\left(J_d = \varepsilon \dfrac{\partial E}{\partial t}\right)$ are all perpendicular to the base of the tube and extend between the lower and upper boundaries as shown in Fig. 12–1. The J_d vectors in Fig 12–1 may for example leave the lower boundary at right angles and make contact with the upper boundary in the same manner. These J_d vectors produce a magnetic field within the interior of the guide as indicated by the dashed lines and establish conduction current densities in the walls of the guide as indicated by the J_c lines shown in Fig. 12–1. The various time-phase relationships cannot be shown in a diagram of this

kind, and neither can the space densities of the electric and magnetic fields be depicted. These quantities can however be determined from Maxwell's equations as is shown presently. Figure 12-1 serves the purpose of showing the reader that the interior of the guide is filled with J_d vectors $\left(\text{established by } \varepsilon \frac{\partial E}{\partial t}\right)$ and that these current-density vectors are rendered continuous by conduction current-density vectors, J_c,

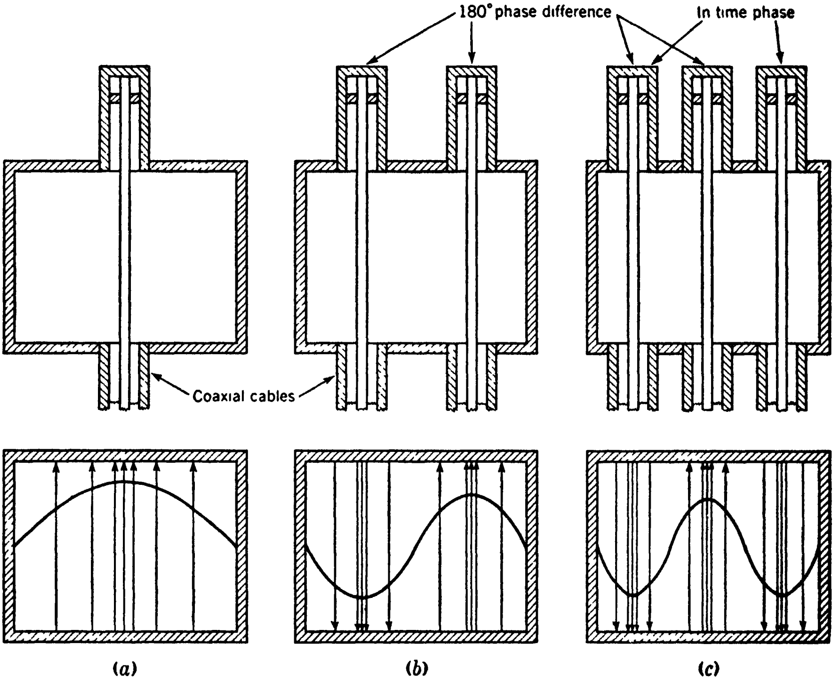


FIG. 12-2. Illustrations of three possible modes of excitation in rectangular wave guides. (a) $TE_{0,1}$, (b) $TE_{0,2}$, (c) $TE_{0,3}$.

which exist in the metal walls of the guide. The precise manner in which the J_d vectors establish a magnetic field within the interior of the guide and the manner in which the E and H fields interact to produce wave propagation along the axial length of the guide will follow a solution of Maxwell's equations which are applicable to the particular boundary conditions employed.

For the present, we shall assume that the J_d configuration shown in Fig. 12-1 may be established in a rectangular wave guide and that in this configuration E_d (which establishes J_d) varies as a half sinusoid from one vertical wall to the other. (Various methods of excitation are

shown in Fig. 12–2.) In any case, E_d must be essentially equal to zero at either vertical wall, and, if the guide is excited in such a manner that maximum E_d occurs at the mid-section, then the half sinusoid of space variation of E_d will fit the boundary conditions. Later it is shown that this space variation of the E_d vectors will satisfy Maxwell's equations and hence represent a possible mode of operation in the rectangular wave guide.

Since $J_c = E_c/\rho = gE_c$, the electric-field intensity E_c in the metal wall is negligibly small if g (the conductivity of the metallic walls) is sufficiently high. Unless the RI^2 power loss in the walls of the guide is

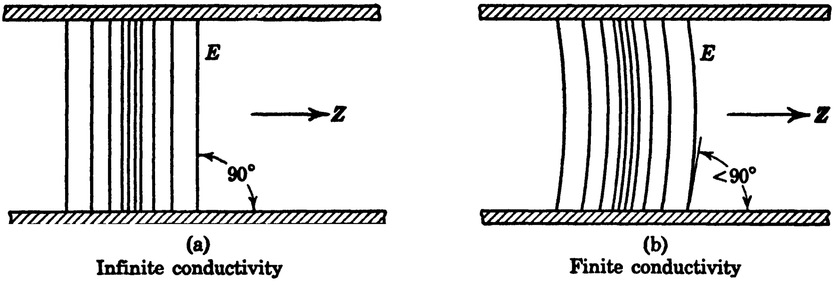


FIG. 12–3. Effect of guide conductivity on shape of electric lines. Propagation in z direction Rectangular TE mode.

of immediate concern, it is customary to assume that E_c is equal to zero, and this assumption is equivalent to assuming $g = \infty$.

If $g = \infty$, the electric vectors meet the top and bottom walls of the guide at precisely 90° as shown in the side view of the guide in Fig. 12–3a. If however, g is finite, a small value of E_c exists at both the top and bottom walls which results in a slight tilt of the E vectors at these walls as shown in Fig. 12–3b. For good conductors, however, this tilt is so slight that it is neglected except where the power loss in the walls is under consideration.

Wave guides may be excited in many different ways with the result that many modes of wave propagation may exist in the guide.[4] In Fig. 12–2 are shown three different modes of excitation of rectangular wave guides. In each case the electric intensity vector is introduced into the guide along a conductor, which is an extension from the output of the

[4] It should be recognized that any field configuration which represents a solution of Maxwell's equations and at the same time satisfies the boundary conditions imposed by the metallic walls of the guide represents a possible mode of operation of the guide. The following chapters are devoted to the details involved in establishing particular modes of operation, and much of what is said here about wave guides is established on a more rigorous basis in these chapters.

oscillator employed to energize the system, in such a manner as to make the E field transverse relative to the axial length of the guide. (See Fig. 12–4.) The intensity of the electric field varies sinusoidally in space, and of the various number of possibilities three are shown in Fig. 12–2. The designations employed to distinguish these three modes of excitation are $TE_{0,1}$, $TE_{0,2}$, and $TE_{0,3}$ where the TE indicates that the electric field is always transverse relative to the direction of propagation, that is, relative to the axial length of the guide. The subscripts 0

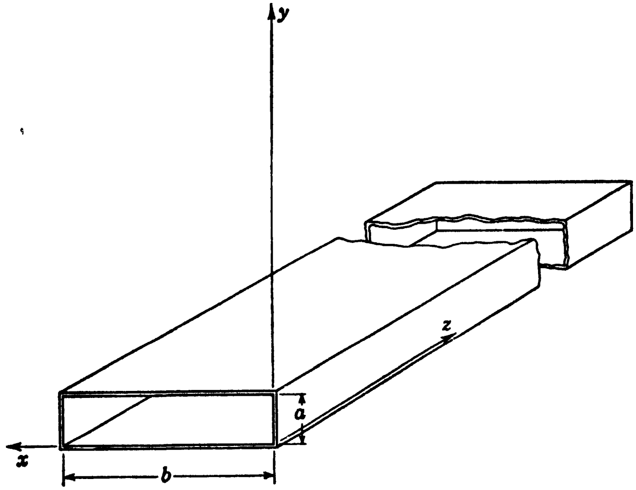


FIG. 12–4. Rectangular wave guide.

indicate that zero space variation of the E vectors occur along the y axis of Fig. 12–4, $\partial E/\partial y = 0$; and the subscripts 1, 2, and 3 refer to the number of half sinusoids of space variation of E which occur between $x = 0$ and $x = b$ of Fig. 12–4. (More elaborate schemes of notation are sometimes employed, but where only a few of the more basic modes of operation are to be considered the scheme employed here appears to be satisfactory.)

In Fig. 12–2a the electric intensity is so distributed that it is maximum in the middle at $x = b/2$ and essentially equal to zero at the side boundaries ($x = 0$ and $x = b$) because of the high conductivity of the walls.

In Fig. 12–2b the boundary conditions are still fulfilled, but the space variation of the electric field intensity goes through one complete cycle from zero at $x = 0$ to zero at $x = b/2$ and to zero again at $x = b$. This mode of operation is induced into the guide by injecting an E_y at $x = b/4$ which is equal in magnitude but opposite in direction to the E_y which is injected at $x = 3b/4$.

The mode of excitation illustrated in Fig. 12–2c is similar in certain respects to a in that the intensities of both electric fields are zero at $x = 0$ and $x = b$ and maximum at $x = b/2$.

In the cylindrical guide shown in Fig. 12–5 it is obvious that the only surface in contact with the electromagnetic field is the inner surface of the guide at the radius $r = b$. At this surface the boundary condition requires that the tangential component of the electric field be zero. The field however may exist in such a tube in a number of different configurations. In any case it will be noted that, if cylindrical coordinates are used as shown, there are two components of the elec-

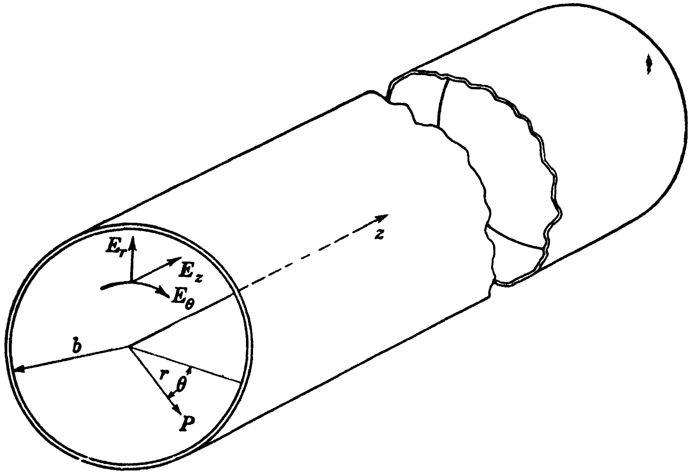


FIG. 12–5. Coordinate system for cylindrical wave guide.

tric field which may be tangential to the inner surface, E_θ and E_z. These components must reduce to zero when $r = b$ if a perfect conductor is assumed. Two possible modes which may exist in such a tube are considered in the following pages. The first mode, shown in Fig. 12–6a, has electric lines which meet the inner surface perpendicularly, E_θ being zero, and which have components E_z in the z direction only at values of r less than b. It is to be noted also that in this mode the field is entirely symmetrical about the axis of the tube. The other mode to be considered is shown in Fig. 12–6b. The electric lines again meet the inner surface perpendicularly; but now, although no E_z component exists, there exist both E_r and E_θ components. It must be kept in mind, however, that at $r = b$, E_θ must be equal to zero. Note that in each of these modes there are no tangential components of the electric field at the boundary.

The mode of operation shown in Fig. 12–6*a* is designated as TM since the magnetic field is transverse relative to the axis of the tube and the mode indicated in Fig. 12–6*b* is designated as TE since the electric field

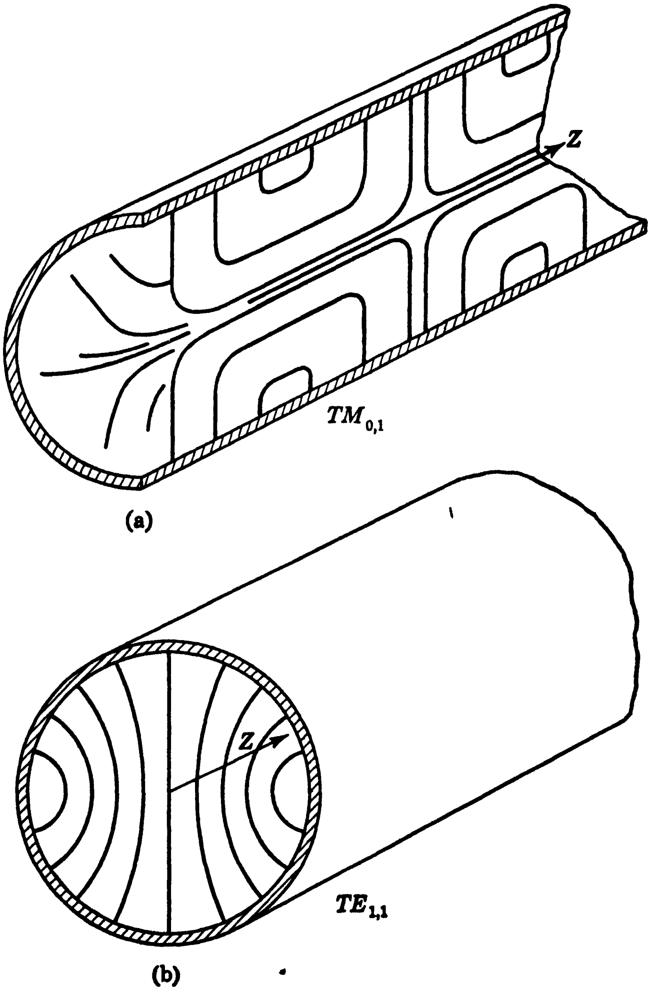


FIG. 12–6. Configuration of electric lines in $TM_{0,1}$ and $TE_{1,1}$, waves.

is now transverse. The subscripts employed in connection with cylindrical wave guides have meanings which are somewhat different from those used in connection with rectangular guides. (See page 297.)

104. Field Configuration and Propagation. Whether one mode or another will actually exist depends on a number of factors to be con-

sidered later. The immediate question concerns the configuration of the magnetic lines which must occur in conjunction with the electric lines. It is necessary to make use of the previously expressed requirement that magnetic lines are everywhere perpendicular to electric lines, and it must also be recalled that magnetic lines must be continuous.

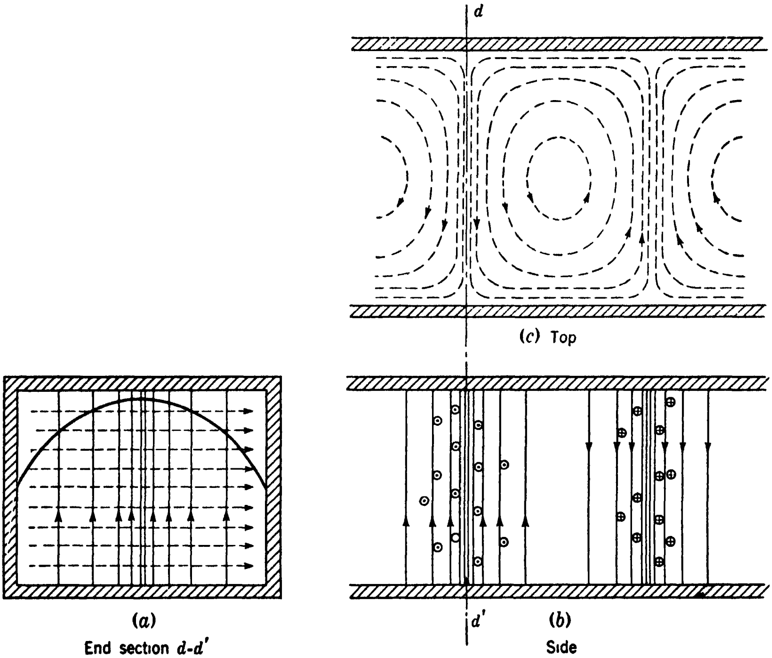


FIG. 12–7. Field configuration in rectangular guides, $TE_{0,1}$ wave.
——— Electric lines of force.
- - - - - Magnetic lines of force.
The circles with crosses and dots represent direction of magnetic lines according to the usual convention.

Let the rectangular $TE_{0,1}$ mode be considered first. It was seen that the only existing electric field component is E_y. This fact immediately requires that the magnetic lines be confined to the x–z plane, or in other words the magnetic field can have no y component. Since the magnetic lines must be continuous, they then exist as closed curves parallel to the x–z plane, such as shown in Fig. 12–7c. An inspection of Fig. 12–7a will show that the electric and magnetic lines of force are represented as having definite directions shown by arrows. Reference to the meaning of the Poynting vector as established in the vector method of handling field equations will establish the following facts.

The flow of power in an electromagnetic field at any point will be given by the equation

$$p = E \times H \quad \text{watts/sq m}$$

or, if ϕ is the space angle between E and H, the value of p may be written

$$p = |EH| \sin \phi \quad \text{watts/sq m}$$

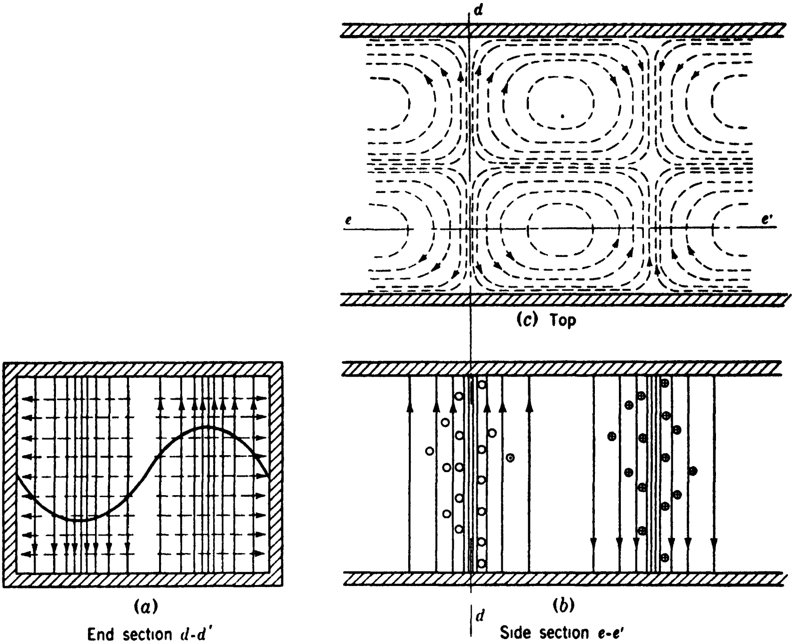


FIG. 12–8 Field configuration in rectangular guides, $TE_{0,2}$ wave. Corresponds to type of excitation shown in Fig. 12–2*b*.

——— Electric lines of force.

- - - - - Magnetic lines of force.

The circles with crosses and dots represent direction of magnetic lines according to the usual convention.

Here ϕ is the smaller space angle between E and H, and the direction of propagation will be given by the right-hand rule applied as follows: Let the curled fingers of the right hand lie in the plane of E and H and point in the direction *from E to H through the smaller angle*; then the thumb will point in the direction of power flow. Applying this rule to Fig. 12–7*a*, using the directions of E and H as shown, it is found that the direction of propagation is along the positive z axis. In Figs. 12–7*b* and 12–7*c* the propagation is toward the right.

The rectangular mode $TE_{0,2}$ is similarly portrayed in Fig. 12–8.

Of the cylindrical modes to be considered, that illustrated in Fig. 12–6*a*, known as the $TM_{0,1}$, allows of only one component of the magnetic field. It is clearly seen that with the electric lines as shown, if the magnetic lines are to be everywhere perpendicular to the electric

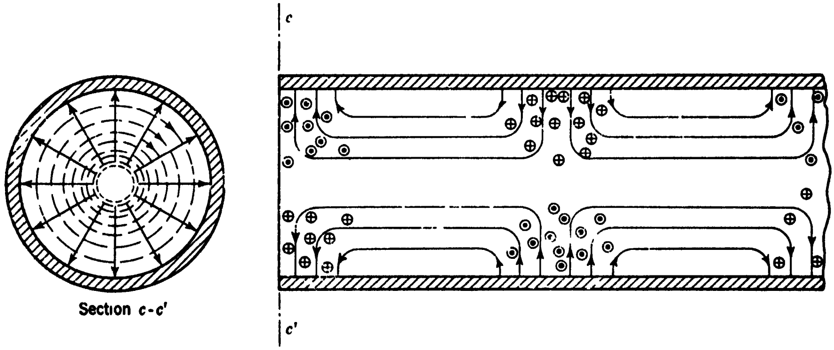


FIG. 12–9. Field configuration in cylindrical guides, $TM_{0,1}$ wave.
——— Electric lines of force.
- - - - - Magnetic lines of force.
The circles with crosses and dots represent direction of magnetic lines according to the usual convention.

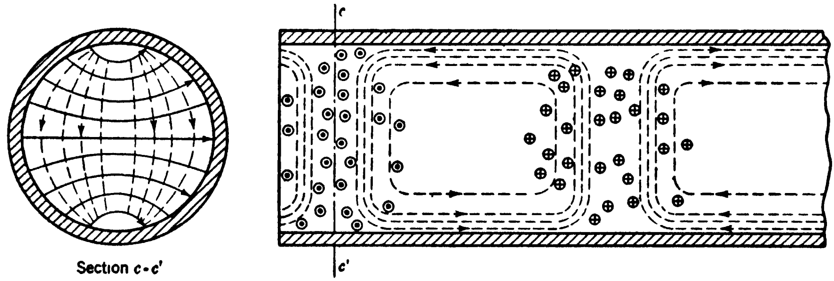


FIG. 12–10. Field configuration in cylindrical guides, $TE_{1,1}$ wave.
——— Electric lines of force.
- - - - - Magnetic lines of force.
The circles with crosses and dots represent direction of electric lines according to the usual convention.

lines, they must then lie in concentric circles about the axis. In other words only the H_θ component can be present, as illustrated in Fig. 12–9. Note here again that the directions of the transverse components of the field are such that the propagation takes place to the right or in the z direction.

The necessary configuration of the magnetic lines in the $TE_{1,1}$ cylindrical mode is not so easy to visualize. However, a careful inspection of Figs. 12–6*b* and 12–10 will show that the magnetic lines must consist

of closed loops which lie parallel to the z axis and which are curved appropriately about this axis in order to be everywhere perpendicular to the electric lines. It is thus seen that the magnetic field in this mode of transmission must have all three possible components, H_r, H_θ, and H_z. In Fig. 12–10 it is again seen that the directions of the fields are such as to produce propagation in the z direction.

105. Determination of Tube Sizes. In this and the following sections, in order to present a brief outline of the physical aspects of wave guides as an introduction to later theoretical work, certain equations from future chapters will be used without immediate discussion of their derivation. These equations are intended to show some of the characteristic features of wave-guide transmission which will be derived in some detail in later chapters.

Wave guides, in general, act like high-pass filters transmitting only frequencies above a certain critical or " cut-off " value. For this reason, as will be seen, they are suitable, in reasonable sizes, only for the transmission of extremely high frequencies. Fortunately the equations giving this cut-off frequency in terms of tube size or vice versa are very simple.

The discussion will first consider the rectangular $TE_{0,1}$ mode presented in Fig. 12–7. It is shown in Chapter XIII that the cut-off frequency for this mode depends only on the width b of the tube and for an air dielectric is given by

$$f_0 = \frac{c}{2b} \text{ cycles/sec} \qquad [13\text{–}19]$$

where c and b are respectively the velocity of light and the tube width in similar units, usually centimeters per second and centimeters, or meters per second and meters. Note that the dimension b must be the one which is perpendicular to the electric lines of the field in the simpler modes. When written in terms of the wavelength of the vibration in air this equation becomes

$$\lambda_0 = c/f_0 = 2b$$

which clearly shows that the critical or cut-off wavelength is that which will just fit into a space twice as wide as the tube. Longer wavelengths than this will not be transmitted.

As an illustration let it be required to find the cut-off frequency and wavelength for a tube which is 10 centimeters wide. From equation 13–19,

$$f_0 = \frac{3 \times 10^{10}}{2 \times 10} = 1.5 \times 10^9 \text{ cycles/sec}$$

The corresponding wavelength is of course 20 centimeters. Frequencies above 1.5×10^9 cycles per second will be transmitted, and for

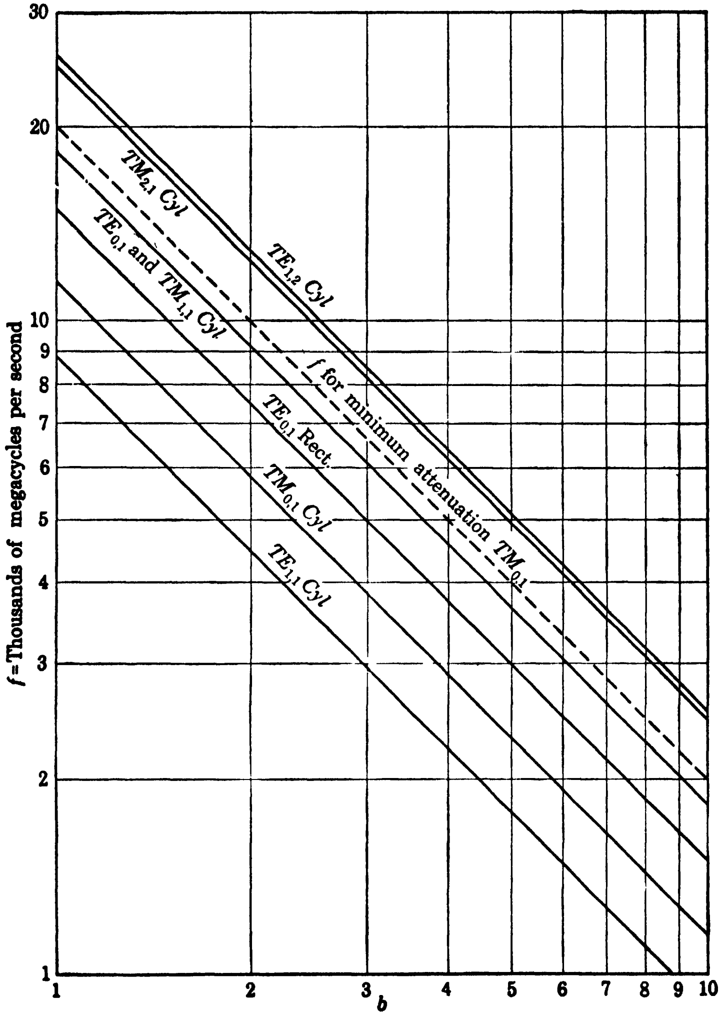


FIG. 12–11. Cut-off frequencies for various modes of transmission.

b = radius of tube in centimeters
b = width of tube in centimeters for rectangular guide

practical work a frequency considerably above this value should be used. Equations for cut-off frequencies of various other modes are as follows.

For the $TM_{0,1}$ mode in cylindrical guides (Fig. 12–9),

$$f_0 = \frac{cp}{2\pi b} \quad \text{cycles/sec} \tag{14–20}$$

where c is the velocity of light, b is the radius of the tube, and $p = 2.405$.

For the $TE_{1,1}$ mode in cylindrical guides (Fig. 12–10) the equation is the same but with $p = 1.84$.

Curves showing the relations between tube dimensions and cut-off frequencies are given in Fig. 12–11 for several different modes of operation. These are plots of equation 14–20 using the appropriate value of p. Parenthetically, it should be noted that this equation applies also to the rectangular tube, for the $TE_{0,1}$ mode, where b is the width of the tube, if p is taken as π. One curve is also given in Fig. 12–11 which represents the frequency for minimum attenuation of the $TM_{0,1}$ mode as a function of b. The curves for f_0 are such that only combinations of f and b represented by points *above* the curves result in transmission. For example, a tube is specified as having a radius of 4 centimeters. What modes will be transmitted at 4×10^9 cycles per second? The answer is immediately seen to be $TE_{0,1}$ rectangular, $TM_{0,1}$, and $TE_{1,1}$.

The following tabulation gives the values of p to be used in the general equation for cut-off frequency:

p	Mode (cylindrical)
1.84	$TE_{1,1}$
2.405	$TM_{0,1}$
3.83	$TE_{0,1}$
3.83	$TM_{1,1}$
5.14	$TM_{2,1}$
5.33	$TE_{1,2}$
π	$TE_{0,1}$ (rectangular)

106. Criteria for Selection of Shape and Size of Tube. The question now arises on what basis the type of tube to be used as well as the mode of excitation is selected. In the consideration of this matter it is necessary to note that a usable tube should be somewhat larger than the critical size for the frequency in question. Also, for most modes there is a frequency at which minimum attenuation occurs, and this frequency is generally a considerable distance beyond f_0. In the rectangular guide one other peculiarity is to be noted. If a tube having dimensions such that $a = b$ is transmitting a certain frequency which is somewhat above the cut-off frequency, then the tube can transmit the same frequency with the electric lines parallel to the dimension b as well as perpendicular to b. Thus such a tube can transmit the same frequency in two different orientations. Such a condition would cause considerable difficulty in

actual practice since, if detecting equipment were constructed to pick up only the wave which has E parallel to a, and if for some reason a parasitic vibration were set up at right angles to this wave, the detector would not indicate its presence.

The problem of detection also dictates certain requirements in cylindrical tubes. If a detector were inserted through the wall of a cylindrical guide which is transmitting the $TE_{1,1}$ mode, it is seen that an accidental change in the orientation or polarization of the wave would result in a decreased pickup. In fact if the plane of polarization were to rotate 90° no indication at all would be obtained. On the other hand it is noted that the $TM_{0,1}$ mode is circularly symmetrical, and such rotation of the wave would not cause any error in the detector reading.

For transmission through rectangular tubes it has been intimated that a form other than square should be used in order to maintain what may be called the original polarization. There is also a requirement on dimensions fixed by attenuation. The fact can be shown that, for minimum attenuation for various tubes of the same perimeter but various values of a/b when transmitting the $TE_{0,1}$ mode, a/b should be about 1.18 where a is the dimension parallel to the electric field. It will immediately be noted that these dimensions will allow the same frequency to be transmitted with both polarizations, and so the attenuation requirement cannot be utilized. Thus, in spite of an increase in attenuation, a/b is made less than unity and may be made about 0.5 in order to transmit as low a frequency as possible with a minimum of copper. The detailed considerations for size are somewhat involved, and in practical cases, since the transmission is generally for short distances, the primary determining factor may be convenience.

As an illustration, the attenuation for a rectangular copper tube 3 inches by 1.5 inches transmitting at 3×10^9 cycles per second in the $TE_{0,1}$ mode may be calculated to be 5.0 decibels per 1000 feet. This is, of course, far from the ideal conditions. The attenuation for a tube having the optimum a/b ratio of 1.18 and the same cut-off and transmitted frequencies as the above 3-by-1.5-inch tube is 2.96 decibels per 1000 feet. Note that here no attempt has been made to transmit the optimum frequency. That the attenuation can be relatively low for ideal conditions is seen from the fact that this guide of 3 inches width, which has the optimum a/b ratio when transmitting the optimum frequency of 3680 megacycles per second, has an attenuation of 2.13 decibels per 1000 feet.

The tube considered above, which has an attenuation of 5.0 decibels per 1000 feet, is designed to transmit 3000 megacycles per second. It is

interesting, before leaving this portion of the general discussion, to determine how well this tube meets the cut-off requirement. The cut-off frequency is given by equation 13–19.

$$f_0 = \frac{3 \times 10^8}{2 \times 3 \times 0.0254} = 1970 \text{ megacycles/sec}$$

Thus certainly 3000 megacycles per second is passed easily. For a wave with a 90° shift in polarization

$$f_0 = \frac{3 \times 10^8}{2 \times 1.5 \times 0.0254} = 3940 \text{ megacycles/sec}$$

Consequently this mode will not be transmitted, so there is no danger of a wave of the same frequency and mode existing in the tube at right angles to the original. The cut-off frequency for the next higher mode $TE_{0,2}$ is given by $f_0 = c/b$, and use of the relation shows that this mode cannot be transmitted since the cut-off frequency is either 3940 or 7880 megacycles per second depending on orientation.

The above discussion brings out the advantage of the rectangular tube transmitting the $TE_{0,1}$ mode. If a/b is approximately 0.5 or less, the tube will not transmit any other mode or orientation at the given frequency. The disadvantage lies in the fact that the tube is not operating with minimum attenuation.

In order to decide definitely on tube size for a cylindrical guide it is necessary to consider the effect of size on attenuation. For most modes minimum attenuation exists at some definite frequency above cut-off, and this frequency varies for different tube sizes. It is advisable to select a tube size which will give a minimum attenuation at approximately the frequency to be used. The $TE_{1,1}$ wave (Fig. 12–10) has a lower attenuation than the $TM_{0,1}$ (Fig. 12–9). However, for short lengths, this disadvantage of the $TM_{0,1}$ wave is outweighed by its advantage, mentioned above, gained by its circular symmetry. The relation for minimum attenuation in the $TM_{0,1}$ mode is $\lambda/b = 1.5$. Actually however any value from 1 to 2 may be used, as in this region the attenuation remains substantially constant. For the $TM_{0,1}$ wave the attenuation curve is almost flat over a band width of about 2 times the cut-off frequency. In a tube of 4 centimeters radius for example the attenuation limits for such a frequency band are 3.96 and 4.15 db per 1000 feet.

As an illustrative problem, assume that it is required to determine the tube size to transmit 3000 megacycles per second and to prevent

the transmission of all modes above $TM_{0,1}$. The cut-off size is given by equation 14–20.

$$f_0 = 3000 \times 10^6 = \frac{3 \times 10^8 \times 2.405}{2\pi b}$$

from which

$$b = \frac{3 \times 10^8 \times 2.405}{3 \times 10^9 \times 2\pi} = 0.0383 \text{ meter}$$

Thus it is known that a tube larger than one having a radius of 3.83 centimeters is to be used. From the relation $\lambda/b = 1.5$,

$$\frac{\lambda}{b} = \frac{c}{fb} = 1.5$$

$$b = \frac{c}{1.5f} = \frac{3 \times 10^8}{1.5 \times 3 \times 10^9}$$
$$= 0.0667 \text{ meter}$$

Thus for minimum attenuation, the radius of the tube should be 6.67 centimeters. It is now necessary to determine whether any higher modes will be transmitted through the tube. The cut-off size (radius) for the next higher mode, the $TE_{0,1}$, is given by $f_0 = \dfrac{3.83c}{2\pi b}$.

$$b = \frac{3.83c}{2\pi f_0}$$
$$= \frac{3.83 \times 3 \times 10^8}{2\pi 3 \times 10^9} = 0.061 \text{ meter or } 6.1 \text{ cm}$$

Accordingly the $TE_{0,1}$ mode would be transmitted, and it is necessary to reduce the tube size to not more than 6.1 centimeters radius. Let it be reduced tentatively to a radius of 5.8 centimeters, and recheck the size for the optimum attenuation condition.

$$\frac{\lambda}{b} = 1.72$$

This is satisfactory because λ/b falls in the range from 1 to 2. It must be remembered that this tube can also transmit the $TE_{1,1}$ mode at this frequency, so that owing to some irregularity the tube may be transmitting two modes simultaneously. In this respect, rectangular wave guides do not suffer from irregularities or tortuosities to the same extent as circular wave guides. It is for this reason that rectangular wave guides are often selected in place of circular wave guides in actual practice.

107. Combinations of Tubes and Modes. Any practical system of wave guides will involve the problem of selecting tube shapes, sizes, and modes of vibration to be used. For the rectangular tube sufficient material has been covered to justify the selection of a guide having dimension a considerably less than b and transmitting its lowest possible mode, the $TE_{0,1}$. For purposes of ordinary transmission in a fixed system this method would be satisfactory. However it suffers from the lack of mechanical flexibility in that it is difficult to transmit the $TE_{0,1}$ mode through elements which may be changing their orientation with respect to the source. For this reason it would seem to be better to transmit only symmetrical waves through cylindrical guides, but, as stated previously, this permits of the existence of lower modes which will interfere with measurements. A compromise must be made by using the symmetrical mode in the movable sections and changing from one to another whenever necessary. This requires, of course, an easy way of changing from rectangular to cylindrical guides and vice versa.

In such a composite system the indicating or detecting equipment would be attached to the rectangular guides at those points where only one mode can exist and that in the correct orientation. Thus any extra loss due to irregularity in the cylindrical section or in the couplings will show itself through an indicated increase in over-all attenuation.

108. Methods of Excitation. Thus far the discussion has been concerned only with transmission through the guides. Two important additional problems, however, are excitation and detection of the waves. Generally speaking the excitation of any given mode may be accomplished by setting up artificially at some point in a tube either an electric or magnetic field of the correct configuration. As a simple illustration let it be required to construct a transfer section which, fed by a small coaxial line, will set up the $TE_{0,1}$ mode in a rectangular tube. In this tube the electric field is entirely transverse, consisting only of the component E_y. The maximum intensity of this component occurs at the center of the tube where $x = b/2$. Such a field suggests the possibility of extending the inner conductor of the coaxial line transversely through the wave guide parallel to E_y, as shown in Figs. 12–2 and 12–12. The coaxial tuning stub b and the sliding collar a are used to provide a support for the end of the conductor as well as to provide a means for adjusting for maximum transfer of power from coaxial cable to wave guide. The transverse conductor is usually placed about a quarter wavelength from the closed end of the tube in order to provide for reinforcement due to reflection in phase at c. If the end of the tube were adjustable so that the distance from the conductor to the end could be varied at will, a still better match between coaxial cable and wave guide could be obtained.

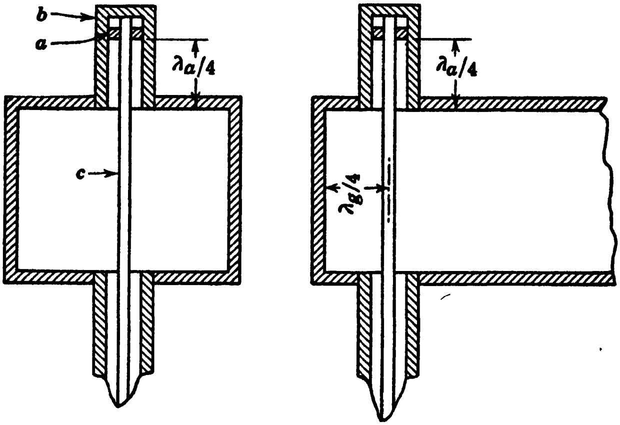


FIG. 12-12. Method of excitation for rectangular $TE_{0,1}$ wave.

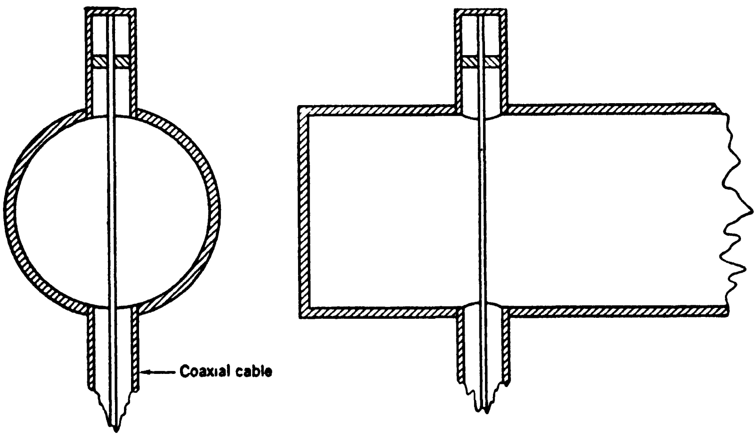


FIG. 12-13. Method of excitation for the $TE_{1,1}$ wave.

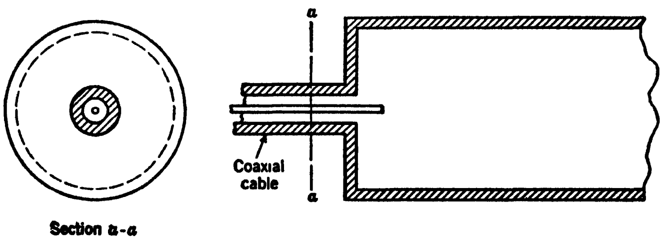


FIG. 12-14. Method of excitation for the $TM_{0,1}$ wave.

On account of the similarity in configuration between the rectangular $TE_{0,1}$ and the cylindrical $TE_{1,1}$ modes the latter may be set up in exactly the same manner as in Fig. 12–12, as shown in Fig. 12–13.

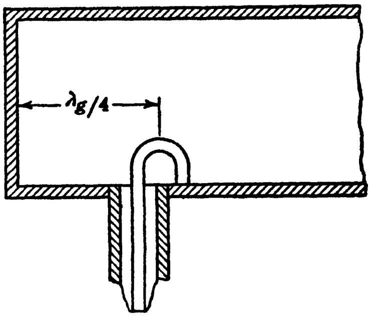


FIG. 12–15. Method of excitation for cylindrical $TM_{0,1}$ wave.

The $TM_{0,1}$ mode has as a characteristic of the field configuration a well-defined axial component of E. The E_z component is a maximum at the center of the tube, and such a field configuration suggests that this mode may be set up by projecting the center conductor of a coaxial cable into the end of the guide as shown in Fig. 12–14. The length of the rod projecting into the end of the tube may be varied in order to obtain optimum transfer of power. Since the only magnetic component in this mode is H_θ, an alternative

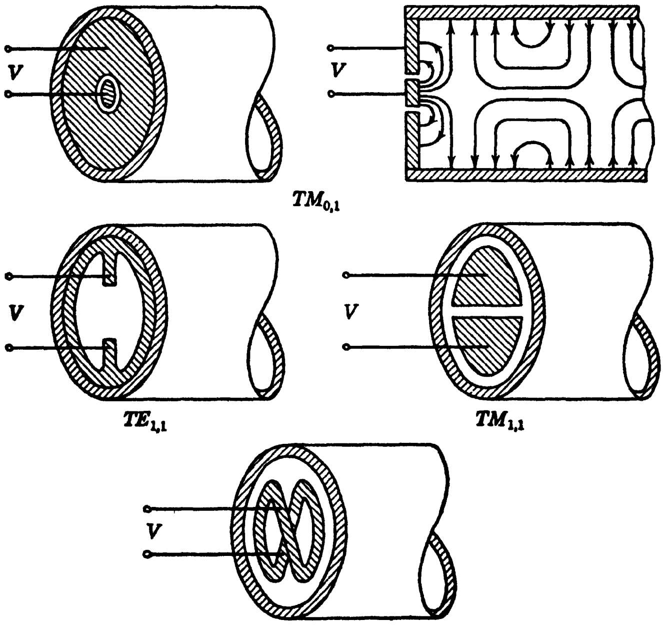


FIG. 12–16. Modes of excitation.

method of excitation is suggested as shown in Fig. 12–15 where a small loop fed by a coaxial line projects into the side of the tube and sets up

the appropriate magnetic field. This is the method which is used in the resonators of some klystrons.

Transfer sections to be used to change from waves guides to coaxial lines are made exactly as in the cases mentioned above. In other words, if a section is good for transmitting from coaxial cable to wave guide, it will also be suitable for the opposite transmission. Alternative methods for exciting various modes in cylindrical guides are presented in Fig. 12–16.

109. Methods of Detection. For detecting the presence of waves in a wave guide the most suitable means available at present is the ordinary

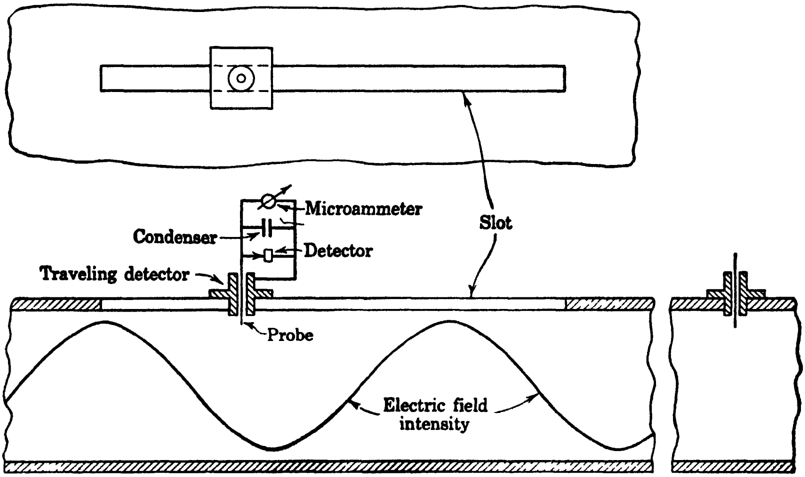


FIG. 12–17. Traveling detector.

crystal detector. The problem of detection is naturally divided into two classes, detection and pickup. It may be necessary merely to detect a wave as it passes along a tube in order to determine the relative amount of power being transmitted or to detect standing waves, etc. In other words it may be necessary to extract only a small amount of the power for purposes of some kind of indication. On the other hand it may be necessary to pick up all the power being transmitted, as for instance when receiving a signal. For the first use mentioned it is essential to disturb the wave in the tube the least possible extent.

The first problem to be considered will be the detection of standing waves in a tube. A so-called traveling detector for either rectangular or cylindrical guides is represented in Fig. 12–17. This detector consists of a fine short wire probe projecting through and insulated from a slide which travels in a longitudinal slot in the guide. For the rec-

tangular guide it is necessary that this detector travel in a side of the guide which is perpendicular to the electric lines of force. It should be centered laterally so that its position is at $x = b/2$ when used to detect the $TE_{0,1}$ mode. In this position the electric intensity is a maximum, and thus the amount of pickup will be greatest. The probe should be as short as possible so as not to disturb the field any more than necessary. Electric lines will end on the probe, thus producing a potential difference between it and the material of the guide near the slot. The probe is connected through a crystal detector to the slide itself, while across the detector is connected a capacitor and a microammeter. This detector

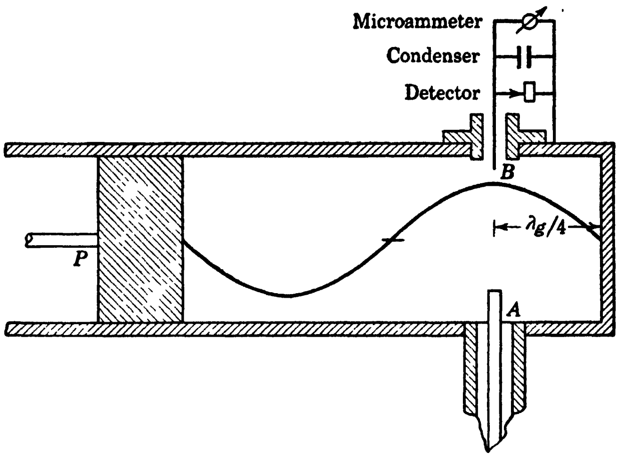


FIG. 12–18. Wavemeter for measuring wavelength in guide.

is known as the shunt type. As the detector is moved along the slide, a pulsating direct current which is a function of the electric-field intensity at the point in question flows through the microammeter. By this means the distribution of the field in the tube can be easily determined. Various mechanical constructions may be used for the traveling detector.

The shunt type of detector is used also in resonant chambers, or wavemeters. One of these is represented in Fig. 12–18. In this instance a small amount of power is introduced into the chamber at A through the coaxial cable. The plunger P is moved until the open space in the tube is some multiple of a half wavelength, whereupon resonance will occur, and the probe and detector at B will indicate a maximum reading. The probe should be placed approximately one-quarter wavelength from the solid end of the chamber. By adjusting the plunger to successive positions which result in maximum readings the wave-

length can be found. It must be remembered that the detectors must always be in the wall on which the electric lines end. This applies to all three modes considered previously. With the $TE_{1,1}$ cylindrical mode, care must be taken that the polarization of the excitation is such that the electric lines meet the tube near the probe. The optimum positions for the detector for the three modes are shown in Fig. 12–19. Note especially that there is no special "best" location of the detector in the $TM_{0,1}$ cylindrical mode. Such a fact agrees with the previous statements as to the advantages of the $TM_{0,1}$ mode. Note that, in the cylindrical $TE_{1,1}$ mode, the detector becomes less effective if for some reason the polarization changes. In particular, if the polarization

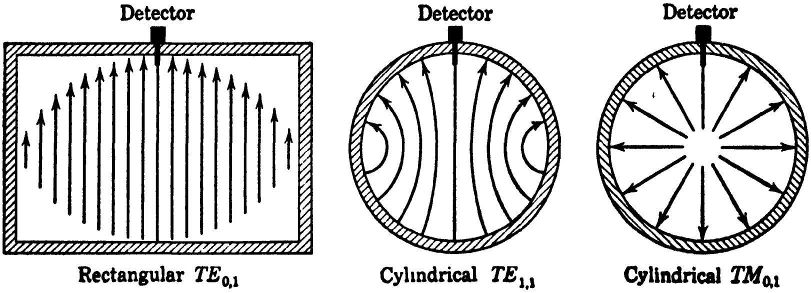


FIG. 12–19. Optimum positions for detector.

rotates by 90°, no power at all can be picked up. In the rectangular $TE_{0,1}$ mode, if the tube were constructed so that the required frequency may be transmitted in both polarizations simultaneously, the detector when placed as shown will indicate only the intensity of the wave which has vertical electric lines.

PROBLEMS

12–1. In Fig. 12–1 assume that at the center of the guide E_d is 50,000 volts per meter of sinusoidal time variation, the frequency of which is 3×10^9 cycles per second. Determine the maximum value of the displacement current density at the center section, assuming that $\epsilon_r = 1$.

12–2. A frequency of 2500 megacycles per second is to be transmitted in a rectangular guide in the $TE_{0,1}$ mode. Determine the minimum dimension of tube width b for such a tube, and suggest a value for the dimension a. Give reasons for the choice of dimension a.

12–3. Find the cut-off frequency for a rectangular tube having dimensions of $a = 1$ cm and $b = 3$ cm when transmitting in the $TE_{0,1}$ mode. Calculate the values of v_p and λ_g in this tube when transmitting at a frequency of $2f_0$.

12–4. Specify the range in size of cylindrical tube which will transmit in the $TE_{1,1}$ mode at 4000 megacycles per second but which will not transmit the $TM_{0,1}$ mode at this frequency.

12-5. Specify the range in size of cylindrical tube which will transmit in the $TM_{0,1}$ mode at 4000 megacycles per second but which will not transmit the $TE_{0,1}$ mode at this frequency.

12-6. A cylindrical copper tube is to be used for the transmission of the $TM_{0,1}$ mode at a frequency of 3500 megacycles per second. What will be the optimum size of tube? Will this tube transmit the $TE_{0,1}$ mode at this frequency?

12-7. It is required to transmit 5000 megacycles per second through a system composed of several feet of rectangular guide operating into a section of cylindrical guide. The rectangular tube is to transmit in the $TE_{0,1}$ mode and the cylindrical tube in the $TM_{0,1}$ mode. Specify the size of rectangular tube and the size of cylindrical tube necessary. Design a coupling for joining the two sections together.

12-8. Suggest a method for connecting a klystron to the rectangular guide of Prob. 12-7 and for transferring the power from the cylindrical guide to a coaxial line. Adjustments should be provided for obtaining the maximum transfer of power.

CHAPTER XIII

ULTRAHIGH-FREQUENCY RECTANGULAR WAVE GUIDES

In the previous chapter the general problem of wave-guide transmission was outlined, and certain equations were used for which no derivations were given. The present chapter is devoted to the mathematical theory of the rectangular wave guide and to the derivation of its fundamental equations. The theoretical derivations begin with the electromagnetic equations developed in Appendix VI.

110. The Differential Equations. Let the problem be proposed to find the characteristics of transmission through a rectangular tube such as shown in Fig. 13–1. The "dominant" mode of transmission which is to be considered is designated by the letters $TE_{0,1}$.[1] The width of the tube is b in the x direction, the depth is a in the y direction, and the longitudinal dimension is along z. The forms of Maxwell's equations as they apply here are equations A–57, A–58, A–59, A–62, A–63, and A–64.

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = gE_z + \frac{\partial \varepsilon E_z}{\partial t} \qquad \text{[A–59]}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = gE_x + \frac{\partial \varepsilon E_x}{\partial t} \qquad \text{[A–57]}$$

[1] See "Electromagnetic Waves in Metal Tubes of Rectangular Cross-Section," by John Kemp, *I.E.E.*, Part 3, Sept. 1941, for an alternative treatment.

The method of designating the modes of transmission is based on the components of the field which are transverse. Thus a field designated by $TE_{n,m}$ has only transverse components of the electric field, in the "dominant" mode E_y only, although in general E_y and E_x may both exist, as well as H_x, H_y, H_z. However, there would be no E_z component. The designation $TM_{n,m}$ means that only transverse components of the magnetic field exist while all components of the electric field may be present.

The subscript notation, as applied to rectangular guides, refers to the number of half wavelengths or maxima of the field-intensity distribution which fit transversely into the guide at the cut-off frequency. The index n indicates the number of half wavelengths found along the y axis and m the number found along the x axis. Thus in the $TE_{0,1}$ mode there is no variation in the electric field along the vertical side, whereas along the base the electric field intensity varies from zero to a maximum and to zero sinusoidally, covering one-half wavelength of this variation. Waves such as $TE_{1,2}$ and $TE_{2,1}$ which have complementary indexes are alike except for orientation in the guide.

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = gE_y + \frac{\partial \varepsilon E_y}{\partial t} \qquad \text{[A-58]}$$

$$\frac{\partial E_y}{\partial x} \quad \frac{\partial E_x}{\partial y} \qquad \frac{\partial \mu_0 H_z}{\partial t} \qquad \text{[A-64]}$$

$$\frac{\partial E_z}{\partial y} \quad \frac{\partial E_y}{\partial z} \qquad \frac{\partial \mu_0 H_x}{\partial t} \qquad \text{[A-62]}$$

$$\frac{\partial E_x}{\partial z} \quad \frac{\partial E_z}{\partial x} \qquad \frac{\partial \mu_0 H_y}{\partial t} \qquad \text{[A-63]}$$

A solution of these equations, as given, would yield propagation in an arbitrary direction in space. In order to simplify matters, since it is

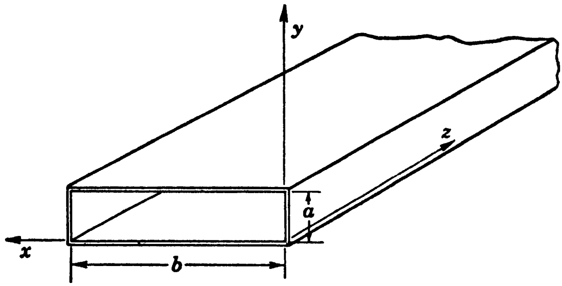


FIG. 13–1. Rectangular wave guide.

known that propagation is to be in the z direction and also that it is desired to treat of only sinusoidal variations in time, some adjustment is necessary. Since only sinusoidal functions are involved, all the time derivatives in the above equations can be eliminated. The terms involving time derivatives will now be transformed as follows: Let H and E be replaced by $\mathcal{H}\epsilon^{j\omega t}$ and $\mathcal{E}\epsilon^{j\omega t}$ respectively where the new $\mathcal{H}$'s and $\mathcal{E}$'s are independent of time and are functions of the space coordinates only.[2]

[2] In the notation of Chapters XII, XIII, XIV, XV, and Appendix VI the light face E and H are general space vectors and are functions of x, y, z, and t. The bold face script $\boldsymbol{\mathcal{E}}$ and script $\boldsymbol{\mathcal{H}}$ are functions of the space coordinates only. The light face script $\mathcal{E}$ and script $\mathcal{H}$ are functions only of the space coordinates *exclusive of z*. Where it is *necessary* to indicate the usual a-c complex vector, which indicates phase position, bold face **E**'s and **H**'s will be used with or without subscripts as the situation may demand. This is a departure from the system used in previous chapters and should be carefully noted.

Wherever the exponential term is used it is to be remembered that only the real part of its trigonometric equivalent is retained. The substitution is justified because

Where no subscript appears with either E or H it is meant that the expression is general and that any subscript desired may be provided. Also that which is given immediately following concerning $\mathcal{E}$ applies also to $\mathcal{H}$. The assumption made above concerning the dependence of E on time amounts to assuming that E is given by the product of two terms: $\mathcal{E}$ and $\epsilon^{j\omega t}$, or

$$\mathcal{E}\epsilon^{j\omega t}$$

where $\mathcal{E}$ is now a function of x, y, and z only. Thus in Maxwell's equations E may be replaced by the above expression, and the derivative $\partial \varepsilon E/\partial t$ may be replaced by $\partial \varepsilon \mathcal{E}\epsilon^{j\omega t}/\partial t = j\omega\varepsilon\mathcal{E}\epsilon^{j\omega t}$. It will be found, as shown in the following paragraph, that the time derivatives will disappear and that the exponential terms may all be divided out.

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of the ease with which the exponential is handled mathematically. Its use may be made clearer by the following brief illustration:

Suppose that a function H is known to be made up of a part dependent sinusoidally on t and a part $\mathcal{H}$ dependent on x, y, and z only. Write

$$H = \mathcal{H}\epsilon^{j\omega t}$$

Remembering that only the real part is to be retained, this is equivalent to

$$H = \mathcal{H} \cos \omega t \qquad \text{[A]}$$

$$= \text{real part of } \mathcal{H} \ (\cos \omega t + j \sin \omega t) \qquad \text{[B]}$$

Now let it be required to find $\partial H/\partial t$. This may be written in two ways, either in the direct manner as from equation A,

$$\frac{\partial}{\partial t}(\mathcal{H} \cos \omega t) \qquad \text{[C]}$$

or as the real part of

$$\frac{\partial}{\partial t}(\mathcal{H}\epsilon^{j\omega t}) \qquad \text{[D]}$$

These will be shown to be equivalent. From C,

$$\frac{\partial}{\partial t}(\mathcal{H} \cos \omega t) = -\mathcal{H}\,\omega \sin \omega t \qquad \text{[E]}$$

From D,

$$\begin{aligned}\frac{\partial}{\partial t}(\mathcal{H}\epsilon^{j\omega t}) &= \mathcal{H}j\omega\epsilon^{j\omega t}\\ &= \mathcal{H}j\omega(\cos \omega t + j \sin \omega t)\\ &= \mathcal{H}\ (-\omega \sin \omega t + j\omega \cos \omega t)\end{aligned}$$

of which the real part is

$$-\mathcal{H}\,\omega \sin \omega t \qquad \text{[F]}$$

which is seen to be the same as equation E.

Let attention be directed to the type of propagation where the only component of the electric field present is $\mathcal{E}_y$. This requires that $\mathcal{E}_x = \mathcal{E}_z = 0$, although all components of $\mathcal{H}$ may be present. The above equations become, after transforming the t derivatives:

$$\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} = 0 \qquad [13\text{-}1]$$

$$\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z} = 0 \qquad [13\text{-}2]$$

$$\frac{\partial \mathcal{H}_x}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial x} = (g + j\omega\varepsilon)\mathcal{E}_y \qquad [13\text{-}3]$$

$$\frac{\partial \mathcal{E}_y}{\partial x} = -j\omega\mu_0\mathcal{H}_z \qquad [13\text{-}4]$$

$$-\frac{\partial \mathcal{E}_y}{\partial z} = -j\omega\mu_0\mathcal{H}_x \qquad [13\text{-}5]$$

$$0 = -j\omega\mu_0\mathcal{H}_y \qquad [13\text{-}6]$$

From equation 13–6 it is seen that $\mathcal{H}_y = 0$.

Let equation 13–4 be differentiated with respect to x.

$$\frac{\partial \mathcal{H}_z}{\partial x} = \frac{-1}{j\omega\mu_0} \cdot \frac{\partial^2 \mathcal{E}_y}{\partial x^2} \qquad [13\text{-}7]$$

From equation 13–5 there is obtained, upon differentiation with respect to z,

$$\frac{\partial \mathcal{H}_x}{\partial z} = \frac{1}{j\omega\mu_0} \cdot \frac{\partial^2 \mathcal{E}_y}{\partial z^2} \qquad [13\text{-}8]$$

If the expressions 13–7 and 13–8 are substituted into equation 13–3, there is obtained

$$\frac{1}{j\omega\mu_0} \cdot \frac{\partial^2 \mathcal{E}_y}{\partial z^2} + \frac{1}{j\omega\mu_0} \cdot \frac{\partial^2 \mathcal{E}_y}{\partial x^2} = (g + j\omega\varepsilon)\mathcal{E}_y \qquad [13\text{-}9]$$

Let this equation be rewritten

$$\begin{aligned} \frac{\partial^2 \mathcal{E}_y}{\partial z^2} + \frac{\partial^2 \mathcal{E}_y}{\partial x^2} &= (-\omega^2\mu_0\varepsilon + j\omega\mu_0 g)\mathcal{E}_y \\ &= h^2\mathcal{E}_y \qquad [13\text{-}10] \end{aligned}$$

where $h^2 = -\omega^2\mu_0\varepsilon + j\omega\mu_0 g$.

The solution of the differential equation 13–10 will give $\mathcal{E}_y$ as a function of x and z. From the value of $\mathcal{E}_y$ the $\mathcal{H}$'s can be determined by using equations 13–4, 13–5, and 13–6.

111. Solution of Equation 13–10. It is to be expected from previous considerations that the propagation of the wave along the tube should give rise to some change in $\mathcal{E}_y$ analogous to that experienced by the voltage vector on a transmission line. In other words, it is safe to assume that part of the solution, that depending on z, is

$$E_{y1} = K\epsilon^{-\gamma z}$$

where γ is the usual propagation constant, written $\gamma = \alpha + j\beta$. Thus

$$\begin{aligned} \mathcal{E}_y &= E_{y1}\mathcal{E}_y \\ &= \mathcal{E}_y K\epsilon^{-\gamma z} \end{aligned}$$

where $\mathcal{E}_y$ is a function of x and y only. Equation 13–10 then becomes

$$\frac{\partial^2}{\partial z^2}\,\mathcal{E}_y K\epsilon^{-\gamma z} + \frac{\partial^2 \mathcal{E}_y}{\partial x^2}\,K\epsilon^{-\gamma z} = (-\omega^2\mu_0\varepsilon + j\omega\mu_0 g)\,\mathcal{E}_y K\epsilon^{-\gamma z}$$

or, differentiating the first term and dividing out $K\epsilon^{-\gamma z}$

$$\gamma^2\mathcal{E}_y + \frac{\partial^2\mathcal{E}_y}{\partial x^2} = (-\omega^2\mu_0\varepsilon + j\omega\mu_0 g)\,\mathcal{E}_y$$

or

$$\frac{\partial^2\mathcal{E}_y}{\partial x^2} = (-\gamma^2 - \omega^2\mu_0\varepsilon + j\omega\mu_0 g)\,\mathcal{E}_y \qquad [13\text{–}11]$$

Since the propagation is taking place in a region where $g = 0$, equation 13–11 may be written

$$\frac{\partial^2\mathcal{E}_y}{\partial x^2} = -k^2\mathcal{E}_y \qquad [13\text{–}12]$$

where $k^2 = \gamma^2 + \omega^2\mu_0\varepsilon$, and $\mathcal{E}_y$ is a function of x and y only.

The solution of equation 13–12 is known to be of the form:

$$\mathcal{E}_y = A_1' \cos kx + A_2' \sin kx$$

It has been assumed that the z term of $\mathcal{E}_y$ is $K\epsilon^{-\gamma z}$ so the complete solution of $\mathcal{E}_y$ becomes

$$\begin{aligned} \mathcal{E}_y &= \mathcal{E}_y E_{y1} \\ &= (A_1' \cos kx + A_2' \sin kx)K\epsilon^{-\gamma z} \\ &= (A_1 \cos kx + A_2 \sin kx)\epsilon^{-\gamma z} \qquad [13\text{–}13] \end{aligned}$$

Two boundary conditions exist which will allow two of the constants, one of the A's and k, to be evaluated. They are

$$\mathcal{E}_y = 0$$

when $x = 0$ and $x = b$

These conditions exist because at the inner surface of the metal tube where the conductivity has been assumed infinite zero-potential difference and hence zero-potential gradient exist. Using these conditions, two equations result

$$0 = [A_1(1) + A_2(0)]\epsilon^{-\gamma z}$$

$$0 = [A_1 \cos kb + A_2 \sin kb]\epsilon^{-\gamma z}$$

From these equations it is easily seen that

$$A_1 = 0$$

$$A_2 \sin kb = 0 \qquad [13\text{–}14]$$

From equation 13–14 it is found that kb must be equal to zero or to some multiple of π radians; that is, $k = m\pi/b$. Making this substitution for k, the solution of equation 13–13 may be written

$$\mathcal{E}_y = A\epsilon^{-\gamma z} \sin\left(\frac{m\pi x}{b}\right) \qquad [13\text{–}15]$$

where the subscript has been dropped from the constant. The above interpretation requires that

$$k = \sqrt{\gamma^2 + \omega^2\mu_0\varepsilon} = \frac{m\pi}{b} \qquad [13\text{–}16]$$

112. Propagation Constant, Cut-Off Frequency, Velocity, Wavelength. Since there are an infinite number of possible values for m, there exists an infinity of solutions for equation 13–15. An important mode of vibration is obtained, however, for the lowest practicable value of m, that is, unity. The wave obtained for this mode of vibration is variously referred to but will be denoted here by the symbol $TE_{0,1}$. For $m = 1$, equation 13–16 provides a means of determining γ the propagation constant. Thus

$$\sqrt{\gamma^2 + \omega^2\mu_0\varepsilon} = \frac{\pi}{b}$$

$$\gamma^2 + \omega^2\mu_0\varepsilon = \left(\frac{\pi}{b}\right)^2$$

from which

$$\gamma^2 = \left(\frac{\pi}{b}\right)^2 - \omega^2\mu_0\varepsilon$$

or

$$\gamma = \sqrt{\left(\frac{\pi}{b}\right)^2 - \omega^2\mu_0\varepsilon} \qquad [13\text{–}17]$$

According to equation 13–17, γ may be either a real quantity or a pure imaginary, depending on the relationship between $(\pi/b)^2$ and $\omega^2\mu_0\varepsilon$. Interest in the present discussion, however, lies only in those waves which are propagated with no attenuation, and thus γ must be a pure imaginary; that is, $\gamma = 0 + j\beta$. Accordingly

$$\gamma = j\sqrt{\omega^2\mu_0\varepsilon - \left(\frac{\pi}{b}\right)^2} = j\beta$$

or

$$\beta = \sqrt{\omega^2\mu_0\varepsilon - \left(\frac{\pi}{b}\right)^2} \qquad [13\text{–}18]$$

If mks values for μ_0, ε, and b are employed, β is expressed in radians per meter length of tube.

The vanishing of α is a direct consequence of the assumption that the material of which the tube is constructed is a perfect conductor. In order for β to be real, it is seen from equation 13–18 that $\omega^2\mu_0\varepsilon$ must be greater than $(\pi/b)^2$. It thus appears that the wave guide will transmit only frequencies above a certain cut-off frequency f_0. The cut-off frequency is obtained from the equality,

$$\omega_0^2\mu_0\varepsilon = \left(\frac{\pi}{b}\right)^2$$

or

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{\mu_0\varepsilon}} \cdot \frac{\pi}{b}$$

The cut-off frequency is

$$f_0 = \frac{1}{2b\sqrt{\mu_0\varepsilon}} \quad \text{cycles/sec}$$

For air, $1/\sqrt{\mu_0\epsilon_0} = c$, the velocity of light.
Therefore

$$f_0 = \frac{c}{2b} \quad \text{cycles/sec} \qquad [13\text{–}19]$$

As an illustration, let it be required to determine the cut-off frequency for a rectangular tube with dimensions of $a = 5$ cm and $b = 8$ cm. It should be noted that no restrictions have thus far been placed on the value of a. For this tube

$$f_0 = \frac{3 \times 10^8}{2 \times 0.08} = 1875 \quad \text{megacycles/sec}$$

The phase-shift constant for this frequency is of course zero, but it increases indefinitely in value as the frequency is increased above the cut-off frequency as indicated by equation 13–18.

The phase velocity is given by

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{b}\right)^2}} \quad \text{meters/sec} \qquad [13\text{–}20]$$

and the wavelength λ_g in the guide is

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{b}\right)^2}} \quad \text{meters} \qquad [13\text{–}21]$$

Let the wavelength of the wave in air, corresponding to this frequency, be written as

$$\lambda_a = \frac{c}{f}$$

$$\frac{\lambda_a}{2\pi} = \frac{c}{\omega}$$

where c is the velocity of light. Substitute this value of c/ω into equations 13–20 and 13–21, and

$$v_p = \frac{\omega}{\sqrt{\left(\frac{2\pi}{\lambda_a}\right)^2 - \left(\frac{2\pi}{2b}\right)^2}}$$

$$= \frac{c}{\sqrt{1 - \left(\frac{\lambda_a}{2b}\right)^2}} \quad \text{meters/sec} \qquad [13\text{–}22]$$

Also

$$\lambda_g = \frac{2\pi}{\sqrt{\left(\frac{2\pi}{\lambda_a}\right)^2 - \left(\frac{2\pi}{2b}\right)^2}}$$

$$= \frac{\lambda_a}{\sqrt{1 - \left(\frac{\lambda_a}{2b}\right)^2}} \quad \text{meters} \qquad [13\text{–}23]$$

As an illustration let it be required to find the phase velocity and wavelength for a wave of frequency $f = 3 \times 10^9$ cycles/sec in a tube of width $b = 10$ cm.

$$\beta = \sqrt{\left(\frac{2\pi 3 \times 10^9}{3 \times 10^8}\right)^2 - \left(\frac{\pi}{0.10}\right)^2}$$

$$= 54.5 \text{ radians/meter}$$

$$v_p = \frac{2\pi 3 \times 10^9}{54.5} = 3.46 \times 10^8 \quad \text{meters/sec}$$

$$\lambda_g = \frac{2\pi}{54.5} = 0.115 \text{ meter} \quad \text{or} \quad 11.5 \text{ cm}$$

The wavelength in air for this wave is

$$\lambda_a = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^9} = 0.10 \text{ meter} \quad \text{or} \quad 10 \text{ cm}$$

The striking result of these calculations is that the velocity in the guide is greater than the velocity of light, and as a consequence the wavelength in the guide is greater than that in air. It is to be noticed that at the cut-off frequency $\beta = 0$ and both v_p and λ_g are infinite.

113. Components of the Magnetic Field. Equation 13–15 shows that $\mathcal{E}_y$ is given as

$$\mathcal{E}_y = A\epsilon^{-\gamma z} \sin \frac{\pi x}{b} \qquad [13\text{–}24]$$

when m has been set equal to unity. A is a constant which depends on the magnitude of the excitation only, and $\gamma = j\sqrt{(\omega/c)^2 - (\pi/b)^2}$. The remaining components of the magnetic field can be determined by means of equation 13–24. Thus, from equations 13–4 and 13–24,

$$\mathcal{H}_z = -\frac{1}{j\omega\mu_0} \cdot \frac{\partial \mathcal{E}_y}{\partial x}$$

$$= \frac{jA\pi\epsilon^{-\gamma z}}{\omega\mu_0 b} \cos \frac{\pi x}{b} \qquad [13\text{–}25]$$

From equations 13–5 and 13–24,

$$\mathcal{H}_x = \frac{1}{j\omega\mu_0} \cdot \frac{\partial \mathcal{E}_y}{\partial z}$$

$$= \frac{jA\gamma\epsilon^{-\gamma z}}{\omega\mu_0} \sin \frac{\pi x}{b} \qquad [13\text{–}26]$$

The component $\mathcal{H}_y$ has already been shown to be zero.

The calculation of the absolute values of the above components can be accomplished as follows: It is necessary to write the terms in their complete form, including the time term. From equation 13–24,[3]

$$E_y = A\epsilon^{-j\beta z}\epsilon^{j\omega t} \sin \frac{\pi x}{b}$$

$$= A\,[\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)] \sin \frac{\pi x}{b}$$

or, taking the real part,

$$E_{y(\text{real})} = A \cos(\omega t - \beta z) \sin \frac{\pi x}{b} \qquad [13\text{–}24\text{a}]$$

In a similar manner, from equation 13–25,

$$H_z = \frac{jA\pi}{\omega\mu_0 b} \epsilon^{-j\beta z}\epsilon^{j\omega t} \cos \frac{\pi x}{b}$$

$$= \frac{jA\pi}{\omega\mu_0 b} [\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)] \cos \frac{\pi x}{b}$$

of which the real part is

$$H_{z(\text{real})} = \frac{-A\pi}{\omega\mu_0 b} \sin(\omega t - \beta z) \cos \frac{\pi x}{b} \qquad [13\text{–}25\text{a}]$$

and, from equation 13–26,

$$H_x = \frac{jAj\beta}{\omega\mu_0} \epsilon^{-j\beta z}\epsilon^{j\omega t} \sin \frac{\pi x}{b}$$

$$= \frac{-A\beta}{\omega\mu_0} [\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)] \sin \frac{\pi x}{b}$$

or

$$H_{x(\text{real})} = \frac{-A\beta}{\omega\mu_0} \cos(\omega t - \beta z) \sin \frac{\pi x}{b} \qquad [13\text{–}26\text{a}]$$

[3] See footnote 2 of Art. 110.

These three equations will give the three components of the field anywhere in the wave guide at any given time. Equation 13–24a, representing the only component of the electric field, indicates, as has already been pointed out, that the electric field for this mode is distributed sinusoidally across the guide. The nonexistence of a magnetic component H_y is consistent with the fact that the electric and magnetic lines must be everywhere perpendicular. In order to arrive at the actual shape of the magnetic lines it is necessary to derive an equation which will give the curves. To do this it is to be noted that the slope of the magnetic lines dx/dz at any point is equal to the ratio of H_x to H_z.

The ratio is found from the equations 13–25a and 13–26a.

$$\frac{dx}{dz} = \frac{H_x}{H_z} = \frac{\beta b}{\pi} \cdot \frac{\cos (\omega t - \beta z) \sin \left(\frac{\pi x}{b}\right)}{\sin (\omega t - \beta z) \cos \left(\frac{\pi x}{b}\right)}$$

Accordingly

$$\frac{\cos \frac{\pi x}{b}}{\sin \frac{\pi x}{b}} dx = \frac{\beta b}{\pi} \cdot \frac{\cos (\omega t - \beta z)}{\sin (\omega t - \beta z)} dz$$

or

$$\cot \frac{\pi x}{b} dx = \frac{\beta b}{\pi} \cot (\omega t - \beta z) dz$$

On integration there is obtained

$$\ln \sin \frac{\pi x}{b} = -\ln \sin (\omega t - \beta z) + \ln C$$

$$\ln \sin \frac{\pi x}{b} + \ln \sin (\omega t - \beta z) = \ln C$$

or

$$\sin \frac{\pi x}{b} \sin (\omega t - \beta z) = C \qquad [13\text{–}27]$$

A set of curves representing the magnetic lines of force in a rectangular guide of 8 centimeters width when transmitting a frequency of 3×10^9 cycles per second is shown in Fig. 13–2. The curves are drawn for the time condition of $t = 0$. The wavelength in the guide is 12.8 centimeters, and the cut-off frequency is 1875 megacycles per second.

The numerical value of the constant C which appears in equation

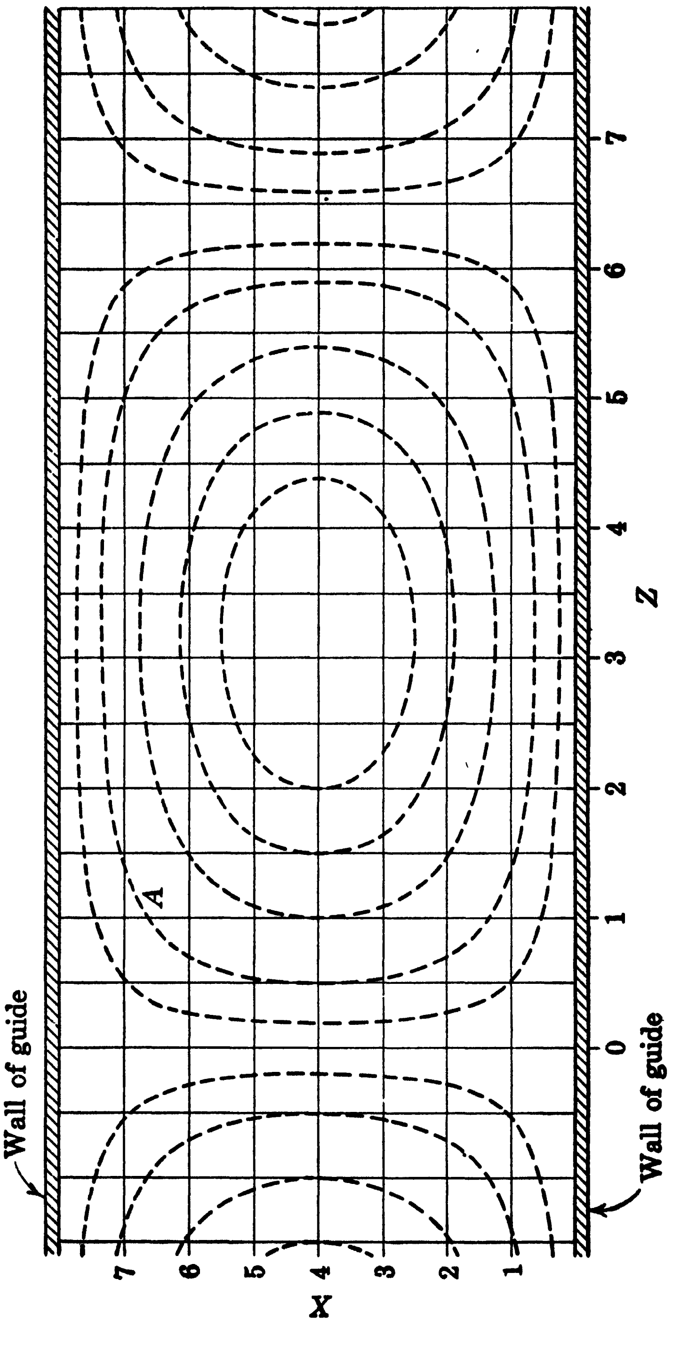

FIG. 13-2. Magnetic lines of force in a rectangular guide. $TE_{0,1}$ mode.

13–27 determines which of the loops is under consideration. The numerical equation for the A loop of the figure is

$$\sin \frac{\pi x}{8} \sin (-0.491z) = -0.243$$

with dimensions taken in centimeters.

114. Calculation of Z_0. Strictly speaking, since there are no conductors involved in the ordinary sense in the wave guide, a characteristic impedance does not exist for it. However, there are voltages and currents involved, as may be seen by the following considerations: An electric-field intensity $\mathcal{E}_y$ exists as given by equation 13–24. A potential difference therefore exists between the top and bottom of the guide. Furthermore, as shown in Appendix VI, it is reasonable to associate a current with H and thus also with $\mathcal{H}$. In fact, H is expressed in amperes per meter. It is also evident that H and the associated guide current should be at right angles to one another. This is clear even from the simple well-known expression $H = I/(2\pi d)$ where H is the field about a wire carrying a current I. Since $\mathcal{E}_y$ is effective from the bottom to the top of the tube, by analogy it would be appropriate to find a current component which is directed along the z axis of the guide. This situation may be visualized to some extent by an inspection of Fig. 13–3. Here the total magnitude of voltage difference between the bottom and top of the guide at $x = b/2$ where $\mathcal{E}_y$ is maximum is obtained from equation 13–24 as

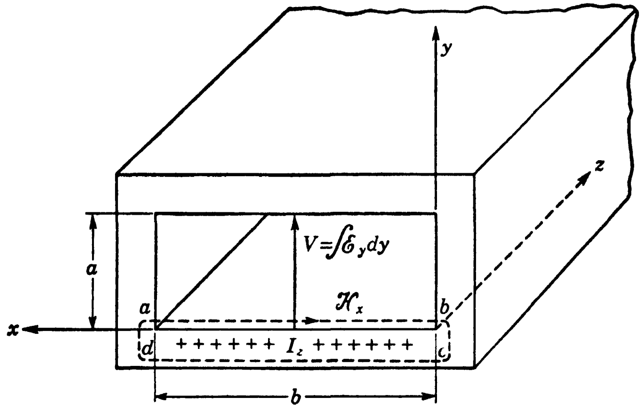

FIG. 13–3. Illustrating the transmission-line analogy of a wave guide.

$$V = \int_0^a \mathcal{E}_y dy = Aa \qquad [13\text{–}28]$$

since $\mathcal{E}_y$ is constant in magnitude at any particular instant of time along the y axis. Since the term $\epsilon^{-\gamma z}$ affects all components the same for a given value of z, it may be omitted here. Also, by the circuital law of magnetism, it is known that the integral of $\mathcal{H}_x$ around the path *abcda* of Fig. 13–3 will give the value of current I_z flowing in the z direction along the lower plate of the guide. The magnitude is given by the integral,

$$I_z = \oint \mathcal{H}_x ds$$

Now at high frequencies it is clear that at any appreciable depth in the metal the electric vector $\mathcal{E}_y$ and the associated magnetic vector $\mathcal{H}_x$ have been reduced effectively to zero, and so in integrating around the path *abcda* it is convenient to notice that only along *a–b* will there be a sensible value of $\mathcal{H}_x$. (See Appendix VI concerning depth of penetration.) Thus the current desired is easily obtained by the following integral:

$$I_z = \int_b^0 \mathcal{H}_x dx$$

If the evaluation of I_z is performed at $z = 0$, it follows from equation 13–26 that

$$\begin{aligned} I_z &= \frac{-A\beta}{\omega\mu_0} \int_b^0 \sin\left(\frac{\pi x}{b}\right) dx \\ &= \frac{A\beta b}{\omega\mu_0\pi} \cos\left(\frac{\pi x}{b}\right)\Bigg]_b^0 \\ &= \frac{2A\beta b}{\omega\mu_0\pi} \end{aligned} \qquad [13\text{–}29]$$

Now it is clear that we have expressions for a maximum transverse potential difference across the center of the guide and for a longitudinal current based on the peak value of H_x. (This is analogous to a transmission line having line conductors which correspond to the bottom and top plates of the guide.) Since the fundamental units of H and E are amperes per meter and volts per meter, we immediately obtain an expression for $Z_0(=V/I)$ from equations 13–28 and 13–29 which is at least the dimensional equivalent of ohms.

$$\begin{aligned} Z_0 &= \frac{V}{I} = \frac{a\omega\mu_0\pi}{2\beta b} \\ &= \frac{\pi\mu_0 a}{2b} \cdot \frac{\omega}{\beta} \end{aligned} \qquad [13\text{–}30]$$

Using equations 13–20 and 13–22, this becomes

$$Z_0 = \frac{\pi\mu_0 a}{2b} \cdot \frac{c}{\sqrt{1 - \left(\frac{\lambda_a}{2b}\right)^2}} \qquad [13\text{–}31]$$

or, expressed in terms of f instead of λ_a, it is

$$Z_0 = \frac{\pi\mu_0 c}{2} \cdot \frac{a}{b\sqrt{1 - \left(\frac{c}{2bf}\right)^2}}$$

$$= \frac{591a}{b\sqrt{1 - \left(\frac{c}{2bf}\right)^2}} = \frac{591a}{b\sqrt{1 - \left(\frac{f_0}{f}\right)^2}} \text{ ohms} \qquad [13\text{–}32]$$

(Refer to Appendix VI for numerical values of g, ε, and μ_0.)

Another possibility for a definition of Z_0 is to define it as the ratio of $E_{y\,\max}$ to $H_{x\,\max}$, a ratio which also has the dimensions of ohms. Employing this concept and using equations 13–24a and 13–26a,

$$Z_0 = \frac{\omega\mu_0}{\beta} \qquad [13\text{–}32a]$$

By way of illustration, this equation will be compared to equation 13–32 for $a = 5$ cm, $b = 10$ cm, and $f = 2f_0$. For equation 13–32,

$$Z_0' = \frac{591 \times 0.05}{0.10\sqrt{1 - \frac{1}{4}}} = 342 \text{ ohms}$$

For equation 13–32a,

$$Z_0'' = \frac{2\pi \times 3 \times 10^9 \times 4\pi \times 10^{-7}}{54.5} = 435 \text{ ohms}$$

In general, there is little agreement among the various definitions of Z_0 as applied to wave guides, and in practice the concept of characteristic impedance is not much used. Instead emphasis is placed on the standing-wave ratio, and correct termination is obtained when a traveling detector, similar to that shown in Fig. 12–17, indicates the same voltage when moved along the guide for a distance of at least a half wavelength.

Illustrative Example. Let it be required to plot f_0 against the guide width b, and β, v_p, λ_g, and Z_0 against the frequency for the 5-by-8-centimeter tube considered in Art. 112. It can be seen from equation 13–19 that the curve relating f_0 and b is a hyperbola. For b expressed in centimeters in this

particular case,

$$f_0 = \frac{15}{b} \times 10^9 \quad \text{cycles/sec}$$

This curve is shown in Fig. 13-4.

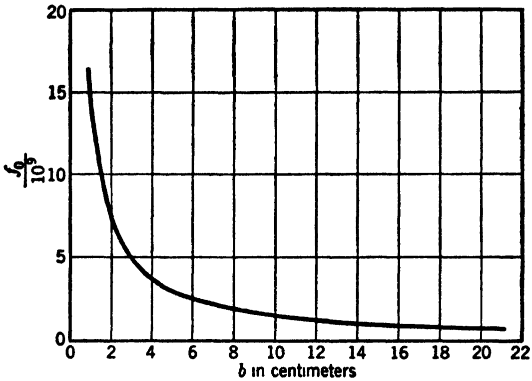

FIG. 13-4. Variation of cut-off frequency with width of rectangular wave guide.

For b expressed in centimeters, the value of β is given by equation 13-18 as

$$\beta = \sqrt{\frac{39.5f^2}{9 \times 10^{20}} - \left(\frac{\pi}{8}\right)^2}$$

$$= \sqrt{4.39f^2 \times 10^{-20} - 0.1542} \quad \text{radians/cm}$$

$$v_p = \frac{2\pi f}{\beta} \quad \text{cm/sec}$$

$$\lambda_g = \frac{2\pi}{\beta} \quad \text{cm}$$

These three quantities are plotted in Fig. 13-5.

It will be noted that the phase velocity in the tube is higher than the velocity of light, only approaching the velocity of light as a limit as the frequency is indefinitely increased.

The characteristic impedance is given by equation 13-32, from which

$$Z_0 = \frac{370}{\sqrt{1 - \frac{0.0351 \times 10^{20}}{f^2}}} \quad \text{ohms}$$

It will be noted that Z_0 is imaginary for frequencies below the cut-off frequency, and at f_0 it becomes infinite. The limiting value of Z_0 as the frequency increases without limit is 370 ohms. The characteristic impedance is plotted in Fig. 13-5.

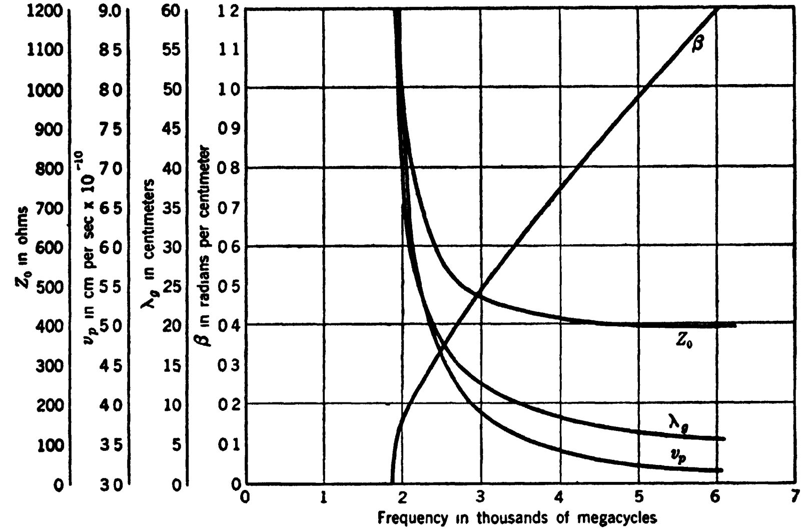

Fig. 13-5. Typical curves for rectangular wave guide.

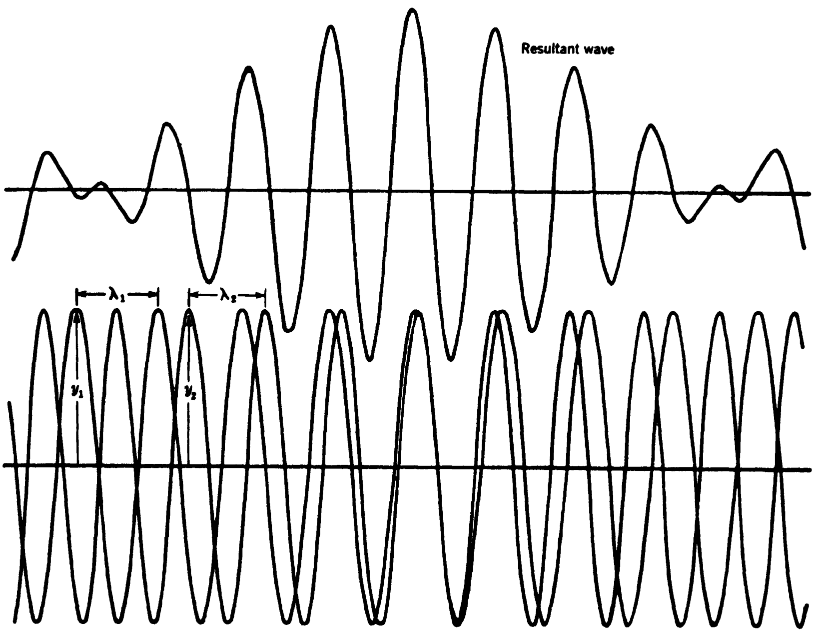

Fig. 13-6. Illustration of group velocity The scale for plotting the resultant wave is not the same as that for waves y_1 and y_2.

115. Group Velocity. Thus far the only velocity considered has been the so-called "phase velocity" which is the velocity of the crest of a wave whose time variation is sinusoidal. This velocity has significance only if one is speaking of a steady-state wave which is already in existence. Obviously, no intelligence can be transmitted by means of this steady-state wave, and in communications of any kind waves must vary in a certain manner and must be finite in duration. The actual transmission of energy then takes place at a velocity known as the group velocity v_g and corresponds to the velocity of the region of reinforcement or interference when two waves, for instance, of slightly different wavelength and velocity are superposed. In wave guides the group velocity is always less than the phase velocity. It will be found that, in the wave guide, although the phase velocity v_p may become infinite, the group velocity is always less than the velocity of light.

Let it be assumed that two slightly different waves exist which may be expressed as follows (see Fig. 13–6):
From equation 13–24a,

$$E_{y(\text{real})1} = A' \cos (\omega_1 t - \beta_1 z)$$

$$= A' \cos 2\pi \left(f_1 t - \frac{z}{\lambda_1}\right) \qquad [13\text{–}33]$$

$$E_{y(\text{real})2} = A' \cos 2\pi \left(f_2 t - \frac{z}{\lambda_2}\right) \qquad [13\text{–}34]$$

The sum of these two waves is

$$E_y = A'\left[\cos 2\pi\left(f_1 t - \frac{z}{\lambda_1}\right) + \cos 2\pi\left(f_2 t - \frac{z}{\lambda_2}\right)\right]$$

$$= 2A' \cos \pi\left[(f_1+f_2)t - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)z\right] \cdot \cos \pi\left[(f_1-f_2)t - \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)z\right] \qquad [13\text{–}35]$$

When $\lambda_1 \doteq \lambda_2$,

$$f_1 + f_2 = 2f \qquad f_1 - f_2 = d(f)$$

and

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{2}{\lambda} \qquad \frac{1}{\lambda_1} - \frac{1}{\lambda_2} = d\left(\frac{1}{\lambda}\right)$$

Then

$$E_y = \left\{2A' \cos \pi\left[d(f)t - d\left(\frac{1}{\lambda}\right)z\right]\right\} \cos 2\pi\left(ft - \frac{z}{\lambda}\right)$$

$$= \left\{2A' \cos \pi\left[d(f)t - d\left(\frac{1}{\lambda}\right)z\right]\right\} \cos \frac{2\pi}{\lambda}(vt - z) \qquad [13\text{–}36]$$

Equation 13–36 is the expression for a wave traveling in the z direction with a phase velocity v and an amplitude given by the bracket term. Interest lies in the determination of the velocity of the crest of the resultant wave. This velocity is given by the condition that the amplitude term must remain at its maximum value, or

$$d(f)t = d\left(\frac{1}{\lambda}\right)z$$

or

$$\frac{z}{t} = v_g = \frac{d(f)}{d\left(\frac{1}{\lambda}\right)} = \frac{d(2\pi f)}{d\left(\frac{2\pi}{\lambda}\right)} = \frac{d\omega}{d\beta} \qquad [13\text{–}37]$$

For the rectangular wave guide,

$$\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{b}\right)^2}$$

and

$$\frac{d\beta}{d\omega} = \frac{\omega}{c^2\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{b}\right)^2}} = \frac{\omega}{\beta c^2}$$

or

$$\frac{d\omega}{d\beta} = \text{group velocity, } v_g = \frac{c^2}{v_p} \qquad [13\text{–}38]$$

An inspection of equation 14–21 shows that equation 13–38 holds also for the cylindrical wave guide Equation 13–38 shows that, since v_p is always greater than the velocity of light, v_g is always less than the velocity of light.

In passing, it is of interest to apply equation 13–37 to the distortionless line treated in Art. 40, wherein $\beta = \omega\sqrt{LC}$. Then

$$\frac{d\beta}{d\omega} = \sqrt{LC}$$

or

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{\sqrt{LC}} = \frac{\omega}{\omega\sqrt{LC}} = \frac{\omega}{\beta}$$

It is thus seen that for this line the group velocity and phase velocity are equal.

116. Attenuation in Rectangular Guides. In the foregoing discussion of wave guides it has been assumed that the walls of the tubes have been perfect conductors and accordingly that at all frequencies above cut-off

the attenuation is zero. Actually guides are constructed of relatively high-conductivity materials such as brass, copper, and silver, and the finite conductivity will of course produce some loss in the guide since there will be some penetration of the field into the metal. (See Appendix VI.)

The procedure for calculating attenuation will be as follows: First, equations 13–24a and 13–26a will be rewritten to include an assumed attenuation term, $\epsilon^{-\alpha z}$, where α is the usual attenuation constant in nepers per meter. These expressions will be combined in Poynting's vector (Art. 104) and integrated over the cross section of the guide. The power loss per unit length of the guide will then be found by differentiating the power with respect to z, and finally the expressions obtained may be rearranged to give an explicit expression for α in terms of normally specified quantities.

Equations 13–24a and 13–26a, rewritten to include the attenuation term, are as follows:

$$E_y = A\epsilon^{-\alpha z} \cos (\omega t - \beta z) \sin \frac{\pi x}{b} \qquad [13\text{–}39]$$

$$H_x = -\frac{A\beta}{\omega\mu_0} \epsilon^{-\alpha z} \cos (\omega t - \beta z) \sin \frac{\pi x}{b} \qquad [13\text{–}40]$$

Since the space angle between these fields is 90° and since the units are volts per meter and amperes per meter, we may write the peak power employing the Poynting-vector concept as

$$p = \frac{A^2\beta}{\omega\mu_0} \epsilon^{-2\alpha z} \cos^2 (\omega t - \beta z) \sin^2 \frac{\pi x}{b} \quad \text{watts/sq m} \qquad [13\text{–}41]^4$$

The time-averaged power per unit cross section of the x–y plane becomes

$$P_{\text{avg}} = \frac{A^2\beta}{2\omega\mu_0} \epsilon^{-2\alpha z} \sin^2 \frac{\pi x}{b} \qquad [13\text{–}42]$$

The total power is now obtained by integrating over the cross section of

[4] This expression is positive because Poynting's vector is written in vector notation as

$$\begin{aligned} p &= E \times H \\ &= jE_y \times iH_x \\ &= -kE_yH_x \end{aligned}$$

which, on substituting equations 13–39 and 13–40, becomes positive. It is noted that the direction of propagation turns out to be $+k$ which means that propagation takes place along the positive z axis as specified by equations 13–39 and 13–40.

the tube opening. This becomes

$$P_t = \frac{A^2\beta a}{2\omega\mu_0}\epsilon^{-2\alpha z}\int_0^b \sin^2\frac{\pi x}{b}\,dx$$

$$= \frac{A^2\beta ab}{4\omega\mu_0}\epsilon^{-2\alpha z} \qquad [13\text{–}43]$$

The power *loss* per unit length is

$$P_L = -\frac{dP_t}{dz} = \frac{2\alpha A^2\beta ab}{4\omega\mu_0}\epsilon^{-2\alpha z} \qquad [13\text{–}44]$$

from which

$$\alpha = \frac{P_L}{2P_t} \qquad [13\text{–}45]$$

This should be compared with equation 5–46. (Note that if the right side of equation 5–46 is multiplied by the square of the line current in both numerator and denominator equation 13–45 is obtained.) The transmitted power at $z = 0$ is given by expression 13–43 as $\dfrac{A^2\beta ab}{4\omega\mu_0}$.

To find P_L note that at the surfaces of the guide we have the following expressions for H. Along the vertical sides such as at A in Fig. 13–7, from equation 13–25a,

$$H_z = \frac{A\pi}{\omega\mu_0 b}\sin(\omega t - \beta z) \qquad [13\text{–}46]$$

At a point such as B, from the same equation,

$$H_z = -\frac{A\pi}{\omega\mu_0 b}\sin(\omega t - \beta z)\cos\frac{\pi x}{b} \qquad [13\text{–}47]$$

At a point such as C, from equation 13–26a,

$$H_x = -\frac{A\beta}{\omega\mu_0}\cos(\omega t - \beta z)\sin\frac{\pi x}{b} \qquad [13\text{–}48]$$

In the preceding material, where the conductivity has been assumed to be infinite, there is no electric wave transmitted into the metal, and thus no power enters the walls. However in this case, where g is finite, an electric field may exist in the metal, and the immediate problem is to determine its magnitude. In changing g from infinity to its actual value for the guide walls, it will be assumed that the resulting percentage

change in the magnetic field at the surface is negligible. Just inside the metal surface there will be an electric field which will be determined through a knowledge of H and the characteristics of the metal. We may define the intrinsic impedance of the metal in terms of $\mathbf{E}/\mathbf{H}$ as follows;[5] where boldface letters are now used because time phase angles enter the picture.

$$\mathbf{Z}_i = \sqrt{\frac{\omega\mu}{g}}\,\underline{/45^\circ}\text{ ohms} \tag{13-49}$$

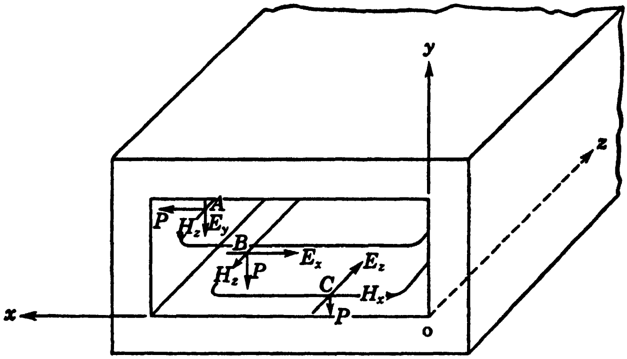

FIG. 13–7. For determining the power loss in the walls of a rectangular wave guide.

where of course μ and g refer respectively to the permeability and the conductivity of the metal walls of the guide. Then the electric fields corresponding to equations 13–46, 13–47, and 13–48 are

$$\left.\begin{aligned} \mathbf{E}_y &= \mathbf{Z}_i\mathbf{H}_z \\ \mathbf{E}_x &= \mathbf{Z}_i\mathbf{H}_z \\ \text{and}\qquad \mathbf{E}_z &= \mathbf{Z}_i\mathbf{H}_x \end{aligned}\right\} \tag{13-50}$$

Equation 13–49 indicates that the magnetic component lags the electric component by 45°, and so, in writing the expression for power density in general,

$$P = E_{\text{rms}}H_{\text{rms}}\cos 45^\circ \tag{13-51}$$

where cos 45° is the power factor and E_{rms} and H_{rms} represent the rms values of the quantities in question.

The rms values of H corresponding to equations 13–46, 13–47, and

[5] See equation A–91, Appendix VI.

13–48 are, respectively,

$$\left.\begin{aligned}
&\text{(At } A\text{)} \qquad && H_{z(\text{rms})} = \frac{A\pi}{\sqrt{2}\omega\mu_0 b} \\
&\text{(At } B\text{)} \qquad && H_{z(\text{rms})} = \frac{A\pi}{\sqrt{2}\omega\mu_0 b}\cos\frac{\pi x}{b} \\
&\text{(At } C\text{)} \qquad && H_{x(\text{rms})} = \frac{A\beta}{\sqrt{2}\omega\mu_0}\sin\frac{\pi x}{b}
\end{aligned}\right\} \qquad [13\text{–}52]$$

The power flowing into the side walls per unit area, such as at A in Fig. 13–7, then becomes, when equations 13–50 and 13–51 are used,

$$\begin{aligned}
P &= |Z_i| H^2_{z(\text{rms})} \frac{1}{\sqrt{2}} \\
&= \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{A^2\pi^2}{2\omega^2\mu_0^2 b^2} \quad \text{watts/sq m} \qquad [13\text{–}53]
\end{aligned}$$

The power flowing into both sides of the guide for 1 meter length, is

$$P' = \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{aA^2\pi^2}{\omega^2\mu_0^2 b^2} \quad \text{watts} \qquad [13\text{–}54]$$

Power flows into the top and bottom of the guide on account of two different field configurations. One is at points like B where $\boldsymbol{H}$ is longitudinal and the other at points like C where $\boldsymbol{H}$ is transverse. Both conditions exist, of course, to some degree, at all points of the top and bottom. At B the power per unit area is as follows, using the second of equations 13–52,

$$\begin{aligned}
P &= |Z_i| H^2_{z(\text{rms})} \frac{1}{\sqrt{2}} \\
&= \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{A^2\pi^2}{2\omega^2\mu_0^2 b^2} \cos^2\frac{\pi x}{b} \quad \text{watts/sq m} \qquad [13\text{–}55]
\end{aligned}$$

This component of power flow into both the bottom and the top of the guide per unit length is given by integrating across the tube and multiplying by 2:

$$\begin{aligned}
P'' &= \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{A^2\pi^2}{\omega^2\mu_0^2 b^2} \int_0^b \cos^2\frac{\pi x}{b}\,dx \\
&= \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{A^2\pi^2}{2\omega^2\mu_0^2 b} \quad \text{watts} \qquad [13\text{–}56]
\end{aligned}$$

Similarly the power flow into top and bottom on account of fields involving H_x, as at C, is

$$P''' = \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{A^2\beta^2}{\omega^2\mu_0^2} \int_0^b \sin^2 \frac{\pi x}{b}\, dx$$

$$= \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{A^2\beta^2 b}{2\omega^2\mu_0^2} \quad \text{watts} \qquad [13\text{–}57]$$

The total power loss per unit length is the sum of equations 13–54, 13–56, and 13–57,

$$P_L = \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{A^2}{\omega^2\mu_0^2}\left[\frac{a\pi^2}{b^2} + \frac{\pi^2}{2b} + \frac{\beta^2 b}{2}\right] \quad \text{watts} \qquad [13\text{–}58]$$

Now from equations 13–43, 13–45, and 13–58 the attenuation constant becomes

$$\alpha = \frac{P_L}{2P_t} = \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{2}{\omega\mu_0\beta ab}\left[\frac{a\pi^2}{b^2} + \frac{\pi^2}{2b} + \frac{\beta^2 b}{2}\right]$$

Substituting

$$\beta^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{b}\right)^2$$

and

$$\lambda = c/f, \quad \lambda_0 = 2b, \quad c = 1/\sqrt{\mu_0\epsilon_0},$$

into the above equation, there is obtained,

$$\alpha = \sqrt{\frac{\omega\mu}{2g}} \cdot \frac{1}{\sqrt{\frac{\mu_0}{\epsilon_0}}\, a\sqrt{1 - \left(\frac{\lambda}{\lambda_0}\right)^2}}\left[1 + \frac{2a}{b}\left(\frac{\lambda}{\lambda_0}\right)^2\right] \quad \text{nepers/meter} \qquad [13\text{–}59]$$

where μ refers to the metal and μ_0 is the value for free space.

PROBLEMS

13–1. A rectangular wave guide has dimensions as follows: $a = 3$ cm, $b = 10$ cm. It is excited in such a way that A of equation 13–24 is unity. γ is assumed to equal $j\beta$. A frequency 50 per cent above the cut-off frequency is to be transmitted. Find the components of the electromagnetic field 12 centimeters from the sending end of the tube, at the point $x = b/4$, and $y = a/2$, when $t = 0$.

13–2. Plot Z_0, v_p, and λ_g against frequency for the wave guide of Prob. 13–1 from f_0 to $3f_0$.

13–3. For the wave guide of Prob. 13–1 plot the wavelength in the guide against the wavelength in air over the range of frequencies f_0 to $3f_0$.

13–4. Plot the group velocity of the guide of Prob. 13–1 against frequency over the range f_0 to $3f_0$.

13–5. Derive the numerical equations for the curves which represent the magnetic lines of force in a rectangular wave guide with dimensions of $a = 5$ cm and $b = 8$ cm when excited in the $TE_{0\,1}$ mode The curves are to pass through the following points

(*a*) $x = 4, z = 0\,2$ (*c*) $x = 4, z = 1\,0$

(*b*) $x = 4, z = 0\,5$ (*d*) $x = 4, z = 1\,5$

(*e*) $x = 4, z = 2\,0$

The guide is transmitting at 3000 megacycles per second, and the time condition is $t = 0$

13–6. Plot the equations derived in Prob 13–5 for the magnetic lines of force for that tube

13–7. A rectangular wave guide has dimensions of $a = 1$ cm and $b = 3$ cm The mode is $TE_{0\,1}$, and the frequency is $1\,5f_0$ Let A in equation 13–24a be 10^5 volts per meter Find the total power transmitted Find Z_0 from equation 13–32 and from equation 13–32a

13–8. A rectangular wave guide is made of copper, and its dimensions are $a = 5$ cm and $b = 10$ cm Find the attenuation in db per meter when transmitting frequencies of $1\,5f_0$, $2\,0f_0$ and $2\,5f_0$ Change the a dimension to 11 8 cm and recalculate the attenuation at each frequency

13–9. Show that the ratio of $a/b = 1\,18$ used in Prob 13–8 produces minimum attenuation.

CHAPTER XIV

ULTRAHIGH-FREQUENCY CYLINDRICAL WAVE GUIDES

117. Transmission in Cylindrical Guides. In Chapter XII a few of the many possible modes of electromagnetic vibration in rectangular guides were briefly mentioned. That it is also possible to set up a great number of modes of vibration in cylindrical guides should be evident. The type of wave produced is determined by the method of excitation, provided the size of the tube will permit the transmission of that particular wave.

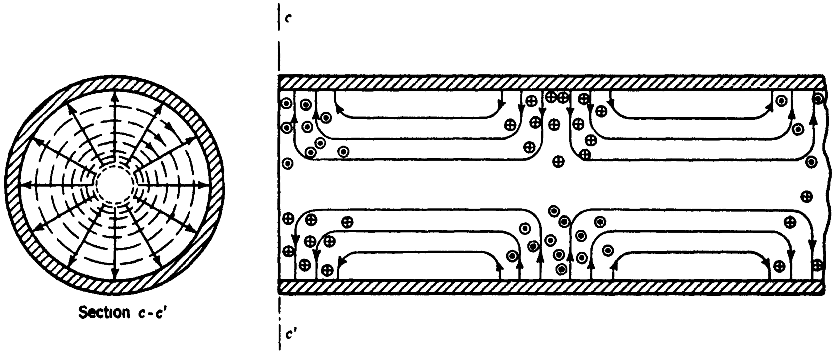

FIG. 14–1. Field configuration in cylindrical guides, $TM_{0,1}$ type.
——— Electric lines of force.
- - - - - Magnetic lines of force.
The circles with crosses and dots represent direction of magnetic lines according to the usual convention.

One of the simplest modes of vibration possible in a cylindrical guide is characterized by magnetic lines which are concentric about the axis of the tube. Such a mode is consistent with the requirement that magnetic lines be continuous; hence both radial and axial components of H are at once eliminated. The electric lines which exist within the guide must meet the requirement that they make contact with the boundary of the tube at right angles, and that they be everywhere perpendicular to the magnetic lines. The tube is assumed to possess infinite conductivity. Thus it is seen that for this particular case the electric field can have components in the direction of propagation and in a radial direction, but in no other directions. In Fig. 14–1 are shown electric (solid)

and magnetic (dash) lines as they exist for this condition. As time passes, the loops of electric lines and the associated magnetic lines progress along the tube at phase velocity. Two loops constitute a wavelength since it is apparent that conditions are repeated after every second loop.

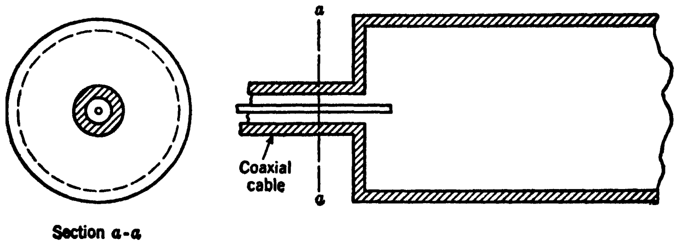

FIG. 14–2. Method of excitation for the $TM_{0,1}$ wave.

The purpose of the present chapter is to outline briefly the theory of this mode of vibration in its simplest form and to derive equations giving its characteristics of transmission. This wave can be obtained by feeding electrostatically an axial rod projecting into one end of the tube as shown in Fig. 14–2. Later, one other mode of transmission in cylindrical guides will be considered, that of the "dominant," or $TE_{1,1}$, wave.

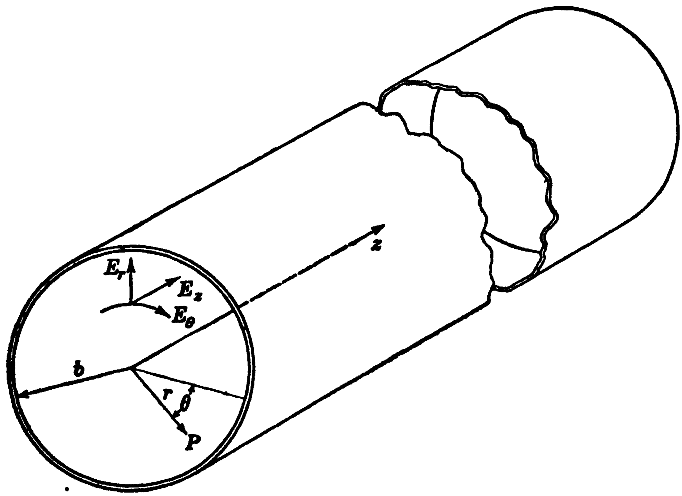

FIG. 14–3. Coordinate system for cylindrical wave guide.

118. The Differential Equations. For transmission through a cylindrical tube the natural coordinates to use are the cylindrical

ones, r, θ, and z, where the tube is oriented so that its axis is along the z coordinate. (See Fig. 14–3.) The Maxwell equations for this case are derived in Appendix VI and are as follows:

$$\frac{\partial H_z}{r\partial\theta} - \frac{\partial H_\theta}{\partial z} = gE_r + \frac{\partial \varepsilon E_r}{\partial t} \qquad \text{[A-67]}$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = gE_\theta + \frac{\partial \varepsilon E_\theta}{\partial t} \qquad \text{[A-68]}$$

$$\frac{1}{r}\left(\frac{\partial rH_\theta}{\partial r} - \frac{\partial H_r}{\partial\theta}\right) = gE_z + \frac{\partial \varepsilon E_z}{\partial t} \qquad \text{[A-69]}$$

$$\frac{\partial E_z}{r\partial\theta} - \frac{\partial E_\theta}{\partial z} = -\frac{\partial \mu_0 H_r}{\partial t} \qquad \text{[A-70]}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial \mu_0 H_\theta}{\partial t} \qquad \text{[A-71]}$$

$$\frac{1}{r}\left(\frac{\partial rE_\theta}{\partial r} - \frac{\partial E_r}{\partial\theta}\right) = -\frac{\partial \mu_0 H_z}{\partial t} \qquad \text{[A-72]}$$

As before, these equations may be simplified by writing $\mathcal{E}\epsilon^{j\omega t}$ in place of E and $\mathcal{H}\epsilon^{j\omega t}$ in place of H, with appropriate subscripts, thus making an assumption that the time variations of E and H are sinusoidal. Also, as before, certain restrictions are placed on the fields in order to obtain a certain type of vibration. Let the fields have components only as follows:[1]

Electric: E_z and E_r
Magnetic: H_θ

that is, $E_\theta = H_r = H_z = 0$. Also, on account of circular symmetry $\partial/\partial\theta = 0$. The conductivity of the dielectric material will again be

[1] This assumption leads to a mode of vibration different from the rectangular $TE_{0,1}$ mode and is introduced here to provide variety of treatment. This mode is designated as $TM_{0,1}$.

The method of designating modes in cylindrical guides is different from that used in rectangular guides. The TM and TE have the same meanings as given in footnote 1 of Art. 110. For the TM mode the first subscript refers to the order of the Bessel function involved in the solution while the second subscript refers to the number of the root of that Bessel function. In the TE mode the first subscript designates again the order of the Bessel function involved, but the second refers to the number of the root of the *derivative* of this Bessel function.

In the $TM_{0,1}$ mode the Bessel function J_0 provides the solution while the first root, 2.4048, is the one used. The $TE_{1,1}$ mode involves the derivative J_1' of the Bessel function J_1 and the first root of J_1, which is 1.84.

considered as zero. Applying these restrictions to equations A-67, A-69, and A-71 there is obtained

$$-\frac{\partial \mathcal{H}_\theta}{\partial z} = j\omega\varepsilon \mathcal{E}_r \qquad [14\text{-}1]$$

$$\frac{1}{r}\left(\frac{\partial r\mathcal{H}_\theta}{\partial r}\right) = j\omega\varepsilon \mathcal{E}_z \qquad [14\text{-}2]$$

$$\frac{\partial \mathcal{E}_r}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial r} = -j\omega\mu_0 \mathcal{H}_\theta \qquad [14\text{-}3]$$

The procedure for setting up the differential equation and its solution follows very closely that given for the solution in Cartesian coordinates.

Write equation 14-2 in the form

$$\frac{r\partial \mathcal{H}_\theta}{\partial r} + \mathcal{H}_\theta = jr\omega\varepsilon \mathcal{E}_z \qquad [14\text{-}4]$$

It will be assumed, in this case also, that the part of the solution dependent upon z, the coordinate in the direction of propagation, is

$$K\epsilon^{-\gamma z}$$

where γ is the propagation constant to be determined. Making this substitution in equation 14-3, in a manner similar to that used for the rectangular guide, there is obtained

$$\gamma \mathcal{E}_r + \frac{\partial \mathcal{E}_z}{\partial r} = j\omega\mu_0 \mathcal{H}_\theta \qquad [14\text{-}5]$$

Equation 14-1 becomes

$$\gamma \mathcal{H}_\theta = j\omega\varepsilon \mathcal{E}_r \qquad [14\text{-}6]$$

and equation 14-4

$$\frac{r\partial \mathcal{H}_\theta}{\partial r} + \mathcal{H}_\theta = jr\omega\varepsilon \mathcal{E}_z \qquad [14\text{-}7]$$

It is desirable to obtain a differential equation expressing a relationship between $\mathcal{E}_z$ and r. To this end first substitute $\mathcal{E}_r$ from equation 14-6 into equation 14-5. From equation 14-6

$$\mathcal{E}_r = \frac{\gamma \mathcal{H}_\theta}{j\omega\varepsilon} \qquad [14\text{-}8]$$

and equation 14-5 then becomes

$$\frac{\gamma^2 \mathcal{H}_\theta}{j\omega\varepsilon} + \frac{\partial \mathcal{E}_z}{\partial r} = j\omega\mu_0 \mathcal{H}_\theta$$

This equation is solved for $\mathcal{H}_\theta$ and the result substituted into equation 14-7.

$$\mathcal{H}_\theta\left(j\omega\mu_0 - \frac{\gamma^2}{j\omega\varepsilon}\right) = \frac{\partial \mathcal{E}_z}{\partial r}$$

or

$$\mathcal{H}_\theta = \frac{-j\omega\varepsilon}{\gamma^2 + \omega^2\mu_0\varepsilon}\cdot\frac{\partial \mathcal{E}_z}{\partial r} \tag{14-9}$$

and

$$\frac{\partial\mathcal{H}_\theta}{\partial r} = \frac{-j\omega\varepsilon}{\gamma^2 + \omega^2\mu_0\varepsilon}\cdot\frac{\partial^2 \mathcal{E}_z}{\partial r^2}$$

Substituting the expressions for $\mathcal{H}_\theta$ and $\partial\mathcal{H}_\theta/\partial r$ into equation 14-7

$$\frac{-j\omega r\varepsilon}{\gamma^2 + \omega^2\mu_0\varepsilon}\cdot\frac{\partial^2 \mathcal{E}_z}{\partial r^2} - \frac{j\omega\varepsilon}{\gamma^2 + \omega^2\mu_0\varepsilon}\cdot\frac{\partial \mathcal{E}_z}{\partial r} = jr\omega\varepsilon\mathcal{E}_z \tag{14-10}$$

Let $k^2 = \gamma^2 + \omega^2\mu_0\varepsilon$. Equation 14-10 may then be written

$$\frac{r}{k^2}\cdot\frac{\partial^2 \mathcal{E}_z}{\partial r^2} + \frac{1}{k^2}\cdot\frac{\partial \mathcal{E}_z}{\partial r} = -r\mathcal{E}_z \tag{14-11}$$

or

$$\frac{\partial^2 \mathcal{E}_z}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial \mathcal{E}_z}{\partial r} + k^2\mathcal{E}_z = 0 \tag{14-12}$$

119. Solution of Equation 14-12. The solution of equation 14-12 is given in Appendix VII, where m^2 appears in the place of k^2 (equation A-108). The solution is $J_0(kr)$, $Y_0(kr)$, or a combination of these such as $H_0^1(kr)$ or $H_0^2(kr)$. Within the tube, where r may become zero, it is necessary to select a solution which will be finite for $r = 0$. Among the above, only $J_0(kr)$ meets this condition. Thus the r part of the solution is

$$\mathcal{E}_z = A'J_0(kr) \tag{14-13}$$

where $k^2 = \gamma^2 + \omega^2\mu_0\varepsilon$. If the z part of the solution be written also there results as a complete solution

$$\mathcal{E}_z = A'K\epsilon^{-\gamma z}J_0(kr) = A\epsilon^{-\gamma z}J_0(kr) \tag{14-14}$$

It is to be noted that $\mathcal{E}_z$ is the component of emf along the axis of propagation; and, since the wall of the tube is assumed to be a perfect conductor, the component $\mathcal{E}_z$ along the wall, at $r = b$, where b is the radius of the tube, is zero. Thus a boundary condition is $\mathcal{E}_z = 0$,

when $r = b$. (See Fig. 14–3.) Therefore from equation 14–14

$$\mathcal{E}_z = A\epsilon^{-\gamma z} J_0(kb) = 0 \tag{14–14a}$$

Accordingly $J_0(kb) = 0$. There are an infinite number of values of kb which will result in a value of zero for $J_0(kb)$. The lowest of these values as found from a table of the function J_0 is 2.4048.[2] Thus

$$kb = b\sqrt{\omega^2\mu_0\varepsilon + \gamma^2} = 2.4048 \equiv p \tag{14–15}$$

The propagation constant, γ, is obtained from equation 14–15.

$$\gamma = \sqrt{\left(\frac{p}{b}\right)^2 - \omega^2\mu_0\varepsilon} \tag{14–16}$$

120. Propagation Constant, Velocity, and Wavelength. The argument concerning the propagation constant for the cylindrical guide follows that given previously for the treatment of propagation in the rectangular guide. Interest lies in those waves transmitted with no attenuation; so

$$\gamma = j\beta = j\sqrt{\omega^2\mu_0\varepsilon - \left(\frac{p}{b}\right)^2} \tag{14–17}$$

or

$$\beta = \sqrt{\omega^2\mu_0\varepsilon - \left(\frac{p}{b}\right)^2} \tag{14–18}$$

If mks values for μ_0, ε and b are employed, β is expressed in radians per meter length of tube. The development now parallels that given for the rectangular wave guide, the only difference being in the fact that here $p(= 2.4048)$ appears instead of π.

The critical frequency which divides the region of transmission from the region of nontransmission is given by $\beta = 0$. Thus

$$\omega_0^2\mu_0\varepsilon = \frac{p^2}{b^2}$$

and

$$\omega_0 = \sqrt{\frac{p^2}{\mu_0\varepsilon b^2}}$$

from which

$$f_0 = \frac{1}{2\pi} \cdot \frac{p}{b\sqrt{\mu_0\varepsilon}} \quad \text{cycles/sec} \tag{14–19}$$

[2] Gray, A., and G. B. Mathews, *Treatise on Bessel Functions*, New York, The Macmillan Co., 1895.

For $f < f_0$ complete attenuation results, and for $f > f_0$ perfect transmission takes place.

If the inside of the tube is air or free space, the substitution $c = 1/\sqrt{\mu_0 \epsilon_0}$ can be made, where c is the velocity of light. The following can then be written:

$$f_0 = \frac{cp}{2\pi b} \text{ cycles/sec} \quad [14\text{–}20]$$

$$\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{p}{b}\right)^2} \text{ radians/meter} \quad [14\text{–}21]$$

$$v_p = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{p}{b}\right)^2}} \text{ meters/sec} \quad [14\text{–}22]$$

$$\lambda_g = \frac{2\pi}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{p}{b}\right)^2}} \text{ meters} \quad [14\text{–}23]$$

121. Illustrative Example. Let it be required to determine f_0, β, v_p, and λ_g for a tube whose diameter is 3 inches. It will be assumed that the material

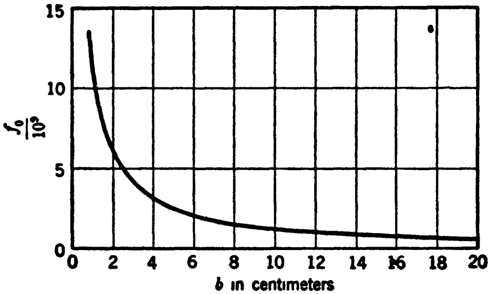

FIG. 14–4. Variation of cut-off frequency with radius of cylindrical wave guide.

is a perfect conductor and that the tube contains air as a dielectric. An equation for f_0 will be written which utilizes the numerical values of c and p, that is,

$$f_0 = \frac{cp}{2\pi b} = \frac{3 \times 10^8 \times 2.4048}{2\pi b_m} = \frac{1.148 \times 10^8}{b_m} \text{ cycles/sec} \quad [14\text{–}24]$$

where b_m is in meters. For the 3-inch tube this becomes

$$f_0 = 3010 \text{ megacycles/sec}$$

Accordingly, the 3-inch tube acts like a high-pass filter which will transmit only frequencies above 3010 megacycles per second. Cut-off frequency

versus tube radius is plotted in Fig. 14–4 for tubes from about 1 to 20 centimeters in radius.

For the 3-inch tube

$$\beta = \sqrt{\frac{\omega^2}{9 \times 10^{20}} - \left(\frac{2.4048}{3.81}\right)^2}$$

$$= \sqrt{4.39f^2 \times 10^{-20} - 0.398} \quad \text{radians/cm}$$

Values of β together with v_p and λ_g are plotted against frequency in Fig. 14–5 for the 3-inch tube.

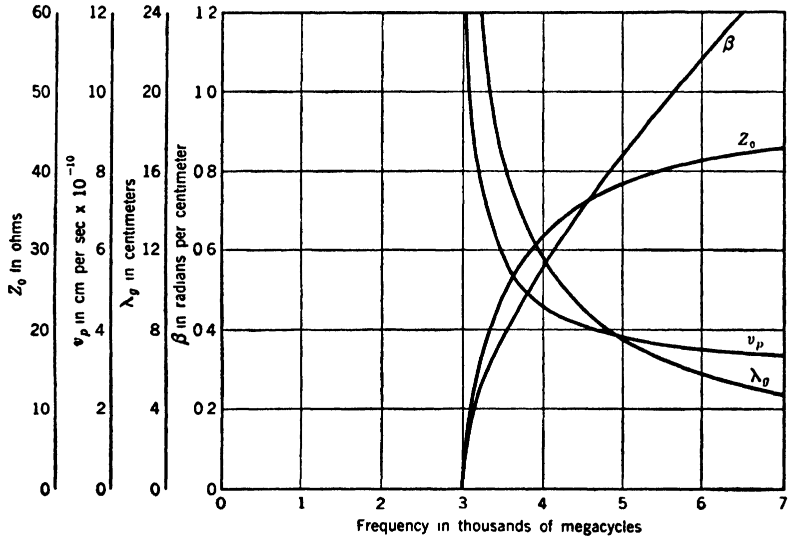

FIG. 14–5. Typical curves for cylindrical wave guide.

122. Calculations of Field Components. One component of the field inside the tube has already been found. Written out completely, including the time term, it is

$$E_z = AJ_0\left(\frac{pr}{b}\right)\epsilon^{j(\omega t - \beta z)} \qquad [14\text{–}25]$$

where p/b has been substituted for k from equation 14–15. Following a procedure similar to that used for the rectangular guide, this component can be written

$$E_z = AJ_0\left(\frac{pr}{b}\right)[\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)]$$

or

$$E_{z(\text{real})} = AJ_0\left(\frac{pr}{b}\right)\cos(\omega t - \beta z) \qquad [14\text{–}26]$$

The component H_θ can be found from equation 14–9, using the expression A–103,

$$\mathcal{H}_\theta = \frac{-j\omega\varepsilon}{\gamma^2 + \omega^2\mu_0\varepsilon} \cdot \frac{\partial \mathcal{E}_z}{\partial r} \qquad [14\text{–}9]$$

$$= \frac{j\omega\varepsilon Ab}{p} J_1\left(\frac{pr}{b}\right) \qquad [14\text{–}9a]$$

and

$$H_\theta = \frac{j\omega\varepsilon Ab}{p} J_1\left(\frac{pr}{b}\right) \epsilon^{j(\omega t - \beta z)}$$

$$= \frac{j\omega\varepsilon Ab}{p} J_1\left(\frac{pr}{b}\right) [\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)]$$

or

$$H_{\theta(\text{real})} = \frac{-\omega\varepsilon Ab}{p} J_1\left(\frac{pr}{b}\right) \sin(\omega t - \beta z) \qquad [14\text{–}27]$$

From equations 14–8 and 14–9a

$$\mathcal{E}_r = \frac{\gamma \mathcal{H}_\theta}{j\omega\varepsilon}$$

$$= \frac{j\beta Ab}{p} J_1\left(\frac{pr}{b}\right)$$

and

$$E_r = \frac{j\beta Ab}{p} J_1\left(\frac{pr}{b}\right) \epsilon^{j(\omega t - \beta z)}$$

$$= \frac{j\beta Ab}{p} J_1\left(\frac{pr}{b}\right) [\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)]$$

or

$$E_{r(\text{real})} = -\frac{\beta Ab}{p} J_1\left(\frac{pr}{b}\right) \sin(\omega t - \beta z) \qquad [14\text{–}28]$$

123. The Field Configuration. It is seen that the field considered in the previous article is made up of one magnetic component, H_θ, which is not a function of θ. Thus the magnetic lines are circles concentric about the tube axis. The shape of the electric lines are found from the slope, dr/dz, of the E curves at any point and this slope must be equal to the ratio of the field intensities at the point in question.

Thus

$$\frac{dr}{dz} = \frac{-\frac{\beta A b}{p} J_1\left(\frac{pr}{b}\right) \sin(\omega t - \beta z)}{A J_0\left(\frac{pr}{b}\right) \cos(\omega t - \beta z)}$$

$$= -\frac{\beta b}{p} \cdot \frac{J_1\left(\frac{pr}{b}\right)}{J_0\left(\frac{pr}{b}\right)} \tan(\omega t - \beta z)$$

This equation can be rearranged to give

$$\frac{J_0\left(\frac{pr}{b}\right)}{J_1\left(\frac{pr}{b}\right)} dr = -\frac{\beta b}{p} \tan(\omega t - \beta z) dz$$

or

$$\frac{\frac{pr}{b} J_0\left(\frac{pr}{b}\right)}{\frac{pr}{b} J_1\left(\frac{pr}{b}\right)} \cdot \frac{p\,dr}{b} = -\beta \tan(\omega t - \beta z) dz^3$$

Integrating both sides of the equation there is obtained

$$\ln\left[\left(\frac{pr}{b}\right) J_1\left(\frac{pr}{b}\right)\right] = -\ln\cos(\omega t - \beta z) + \ln C$$

from which

$$\cos(\omega t - \beta z) = \frac{C}{\left(\frac{pr}{b}\right) J_1\left(\frac{pr}{b}\right)} \qquad [14\text{–}29]$$

A set of curves representing the electric lines of force in a cylindrical guide of 5 centimeters radius when transmitting a frequency of 3×10^9 cycles per second is shown in Fig. 14–6. The curves are drawn for the

[3] A useful differential form in Bessel functions is

$$xJ_0(x) = \frac{d[xJ_1(x)]}{dx}$$

Reddick, H. W., and F. H. Miller, *Advanced Mathematics for Engineers*, New York, John Wiley & Sons, p. 214, 1938.

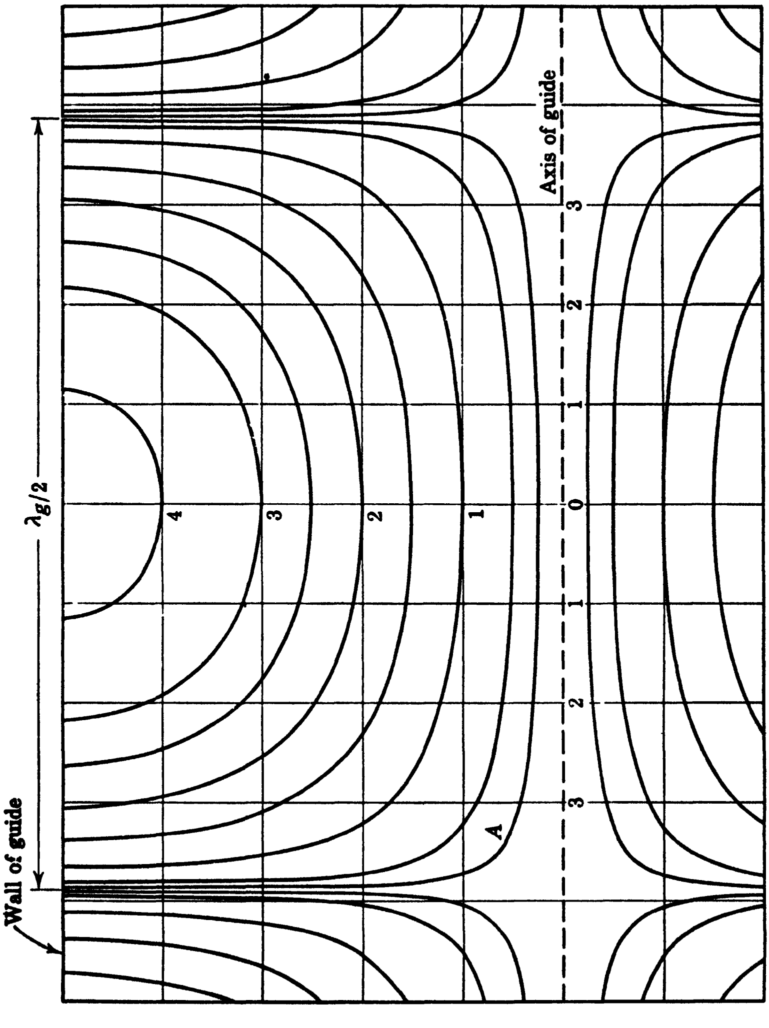

FIG. 14-6. Electric lines of force in cylindrical guide. $TM_{0,1}$ mode.

time condition of $t = 0$. The wavelength in the guide is 15.5 centimeters.

The numerical value of the constant C which appears in equation 14–29 determines which of the curves is under consideration. The numerical equation for the A curve of the figure is

$$\cos(-0.405z) = \frac{0.0072}{(0.481r)J_1(0.481r)}$$

with dimensions taken in centimeters.

124. Characteristic Impedance. The characteristic impedance Z_0 will be defined as the ratio of the transverse emf in the tube to the longitudinal current corresponding to the magnetic field term $\mathcal{H}_\theta$. The transverse emf will be given by

$$V = \int_0^b \mathcal{E}_r dr = Aj\frac{\beta b}{p}\int_0^b J_1\left(\frac{pr}{b}\right) dr$$

where the exponential terms have been omitted since they refer only to the phase of the emf.

$$\begin{aligned} V &= -Aj\beta\frac{b^2}{p^2} J_0\left(\frac{pr}{b}\right)\Bigg]_0^b \\ &= -Aj\beta\frac{b^2}{p^2}[J_0(p) - J_0(0)] \\ &= Aj\beta\frac{b^2}{p^2} \end{aligned} \qquad [14\text{–}30]$$

The current may be obtained by integrating $\mathcal{H}_\theta$ around the inner surface of the tube.

$$\begin{aligned} I &= \oint \mathcal{H}_\theta b d\theta \\ &= \frac{j\omega\varepsilon Ab}{p}(2\pi b)J_1\left(\frac{pb}{b}\right) \\ &= 2\pi j\omega\varepsilon A\frac{b^2}{p}J_1(p) \end{aligned} \qquad [14\text{–}31]$$

where again the exponential terms have been omitted. The characteristic impedance thus becomes

$$\begin{aligned} Z_0 &= \frac{V}{I} = \frac{\beta}{\omega}\cdot\frac{1}{2\pi\varepsilon p J_1(p)} \\ &= 1.442 \times 10^{12} \times \frac{\beta}{\omega} = \frac{1.442 \times 10^{12}}{v_p} \text{ ohms} \end{aligned} \qquad [14\text{–}32]$$

The value of $J_1(p) = J_1(2.4048)$ can be found on reference to a table of Bessel functions and is 0.519. It will be recalled that the quantity ε is $1/(36\pi \times 10^9)$ for free space. (See Appendix VI.)

The limiting value of Z_0 as v_p approaches the velocity of light is 48 ohms. The characteristic impedance as obtained from equation 14–32 is plotted against frequency in Fig. 14–5 for the 3-inch tube.

125. Attenuation in Cylindrical Guides for the $TM_{0,1}$ Mode. Thus far in this chapter it has been assumed that the wave guides are constructed of material having infinite conductivity. Under this condition of course there can be no loss of power, and the attenuation constant is zero above the cut-off frequency. However, as was mentioned in Chapter XIII, if the material has a high but not infinite conductivity, the attenuation is finite though small, and it may be assumed that the tangential magnetic-field intensity next to the metal surface is not appreciably different from the previous value. As in the preceding chapter, it will again be assumed that the attenuation constant α will be given by the expression,

$$\alpha = \frac{1}{2} \frac{\text{power loss per unit length}}{\text{power transmitted}}$$

It is thus necessary to calculate the transmitted power and the power loss per meter length.

The power transmitted may be calculated from equations 14–27 and 14–28. It is noted that these components are perpendicular and in such a direction that the direction of power transmission is along the positive z axis of the tube, according to Poynting's vector.

The magnitude of Poynting's vector is

$$\mathfrak{p} = E_r H_\theta \quad \text{watts/sq m} \qquad [14\text{–}33]$$

Substituting from equations 14–27 and 14–28, the power expression becomes

$$\mathfrak{p} = \frac{\omega\beta\varepsilon A^2 b^2}{p^2} J_1^2\left(\frac{pr}{b}\right) \sin^2(\omega t - \beta z) \quad \text{watts/sq m} \qquad [14\text{–}34]$$

The average value of power is

$$P_{\text{avg}} = \frac{\omega\beta\varepsilon A^2 b^2}{2p^2} J_1\left(\frac{pr}{b}\right) \text{watts/sq m} \qquad [14\text{–}35]$$

In order to obtain the total power transmitted, equation 14–35 must be integrated over the cross section of the wave-guide opening. The expression is not a function of θ; accordingly, the integration with

respect to θ merely means multiplication by $2\pi r$. Thus the total power becomes

$$P_t = \frac{\omega\beta\varepsilon A^2 b^2 \pi}{p^2} \int_0^b J_1^2\left(\frac{pr}{b}\right) r dr \qquad [14\text{–}36]$$

$$= \left(\frac{f}{f_0}\right)^2 \frac{A^2 \pi b^2}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \cdot \sqrt{1 - \left(\frac{f_0}{f}\right)^2}\, J_1^2(p) \text{ watts} \qquad [14\text{–}37][4]$$

on substituting for β from equation 14–18, using $c = 1/\sqrt{\mu_0\epsilon_0}$ and $\varepsilon = \epsilon_0$ for air.

It is now necessary to evaluate the power loss per meter of the line. The only magnetic component present is H_θ which, at the guide wall where $r = b$, has the value given by equation 14–27,

$$H_\theta = \frac{-\omega\varepsilon A b}{p} J_1(p) \sin(\omega t - \beta z) \qquad [14\text{–}38]$$

It will be assumed that reduction of g of the wall material to its actual value will not affect the value of H_θ. Then just beneath the surface of the metal the value of E_z will be given by

$$E_z = Z_i H_\theta \qquad [14\text{–}39]$$

where

$$Z_i = \sqrt{\frac{\omega\mu_0}{g}} \angle 45^\circ$$

as given in Appendix VI. In the immediately following equations the term μ_0 will be used for both the air in the tube and for the metal which is nonferromagnetic.

It is seen that there is a phase difference of 45° between E_z and H_θ in the metal, and the power flow into the metal must be written as follows, using the quantity $1/\sqrt{2}$ as the power factor,

$$p = \sqrt{\frac{\omega\mu_0}{2g}} \cdot \frac{\omega^2\varepsilon^2 A^2 b^2}{p^2} J_1^2(p) \sin^2(\omega t - \beta z) \qquad [14\text{–}40]$$

The average power is

$$P_{\text{avg}} = \sqrt{\frac{\omega\mu_0}{2g}} \cdot \frac{\omega^2\varepsilon^2 A^2 b^2}{2p^2} J_1^2(p) \text{ watts/sq m} \qquad [14\text{–}41]$$

[4] This integration comes from a useful formula in the *theory of Bessel functions*,

$$\int_0^b J_n^2(kx) x dx = \frac{b^2}{2}\left[J_{n-1}(kx) J_{n+1}(kx) + J_n^2(kx)\right]$$

In the above development, $n = 1$ and $J_0(kb) = 0$.

This is a constant, and in order to obtain the value for 1 meter length of line it is only necessary to multiply by the area of the wall, $2\pi b$. Thus the total power loss becomes

$$P_L = \sqrt{\frac{\omega\mu_0}{2g}} \cdot \frac{\omega^2\varepsilon^2 A^2 b^3 \pi}{p^2} J_1^2(p) \text{ watts} \quad [14\text{–}42]$$

$$= \sqrt{\frac{\omega\mu_0}{2g}} \cdot \left(\frac{f}{f_0}\right)^2 \cdot \frac{\varepsilon}{\mu_0} A^2 b\pi J_1^2(p) \text{ watts} \quad [14\text{–}43]$$

and the attenuation becomes

$$\alpha = \frac{P_L}{2P_t} = \sqrt{\frac{\omega\varepsilon}{2g}} \cdot \frac{1}{b\sqrt{1 - \left(\frac{f_0}{f}\right)^2}} \text{ nepers/meter} \quad [14\text{–}44]$$

$$= \frac{1}{b}\sqrt{\frac{\pi\varepsilon}{g}} \cdot \sqrt{\frac{f}{1 - \left(\frac{f_0}{f}\right)^2}} \text{ nepers/meter} \quad [14\text{–}45]$$

The attenuation is thus seen to depend in a somewhat peculiar manner on the frequency. This will be investigated in the following sections.

126. Curves of Attenuation and Phase Shift, $TM_{0,1}$ Mode. Equation 14–45 may now be written into the more general expression for the propagation constant, remembering that, from equation 14–21

$$\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{p}{b}\right)^2}$$

$$= \frac{\omega}{c}\sqrt{1 - \left(\frac{f_0}{f}\right)^2} \quad [14\text{–}46]$$

$$\gamma = \alpha + j\beta = \frac{1}{b}\sqrt{\frac{\pi\epsilon_0}{g}} \cdot \sqrt{\frac{f}{1 - \left(\frac{f_0}{f}\right)^2}} + j\frac{\omega}{c}\sqrt{1 - \left(\frac{f_0}{f}\right)^2} \quad [14\text{–}47]$$

It can be shown that equation 14–47 is also valid when $f < f_0$.[5] At such frequencies this equation becomes,

$$\gamma = \frac{\omega}{c}\sqrt{\left(\frac{f_0}{f}\right)^2 - 1} + j\frac{1}{b}\sqrt{\frac{\pi\epsilon_0}{g}} \cdot \sqrt{\frac{f}{\left(\frac{f_0}{f}\right)^2 - 1}} \quad [14\text{–}48]$$

[5] See, for example, "Attenuation of Electromagnetic Fields in Pipes Smaller than the Critical Size," by E. G. Linder, *Proc. I. R. E.*, Vol. 30, No. 12, Dec. 1942.

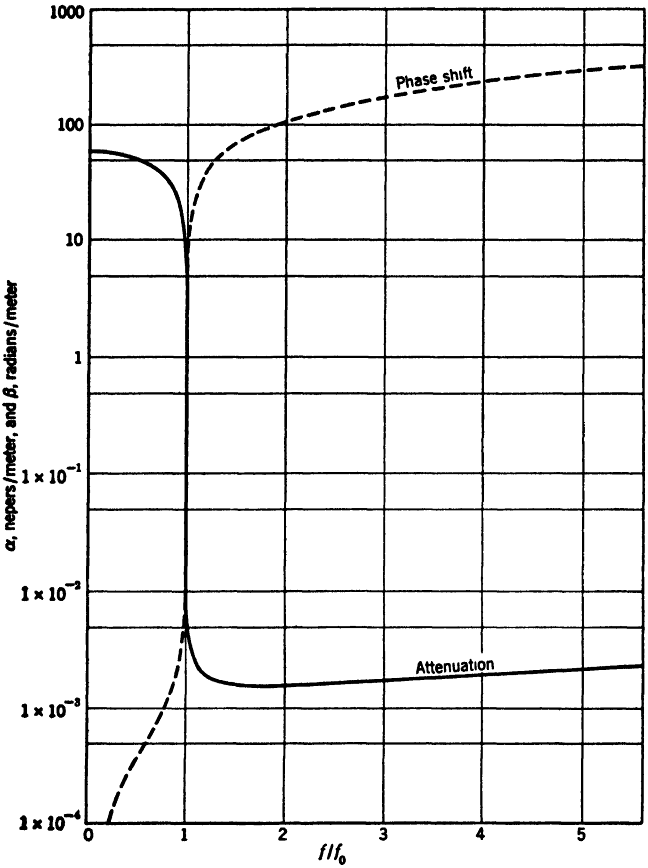

FIG. 14–7. Attenuation and phase shift versus f/f_0 for the cylindrical guide transmitting the $TM_{0,1}$ mode.

from which, for $f < f_0$,

$$\alpha = \frac{\omega}{c}\sqrt{\left(\frac{f_0}{f}\right)^2 - 1} \text{ nepers/meter} \tag{14-49}$$

$$\beta = \frac{1}{b}\sqrt{\frac{\pi\epsilon_0}{g}} \cdot \sqrt{\frac{f}{\sqrt{\left(\frac{f_0}{f}\right)^2 - 1}}} \text{ radians/meter} \tag{14-50}$$

Curves of α and β versus f/f_0 are plotted in Fig. 14–7 for a cylindrical guide of 4 centimeters radius when transmitting in the $TM_{0,1}$ mode.

The constants used are

$$\epsilon_0 = 1/(36\pi \times 10^9) \text{ farad/meter}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry/meter}$$

$$g = 6 \times 10^7 \text{ mho-m/sq m}$$

It should be noted that, for very low values of f, α approaches the value $p/b = 60.1$ nepers/meter. It is interesting to note that, for low values of f, α is independent of tube material and depends only on dimensions. This is the basis for an important type of attenuator used in microwave work.

127. Optimum Size of Wave Guide. On account of the manner of occurrence of f and b in equation 14–45, it might be suspected that some particular combination of these two factors might produce minimum attenuation. In order to investigate this point it is better to work in terms of wavelength; so let equation 14–45 be rewritten as follows, using $f = c/\lambda$, and $f_0 = pc/2\pi b$:

$$\alpha = \frac{\sqrt{\dfrac{\pi c \epsilon_0}{g}}}{b^{3/2}\sqrt{\dfrac{\lambda}{b} - \left(\dfrac{p}{2\pi}\right)^2\left(\dfrac{\lambda}{b}\right)^3}}$$

$$= Kb^{-3/2}M \qquad [14\text{–}51]$$

where

$$K = \sqrt{\frac{\pi c \epsilon_0}{g}}, \quad \text{and} \quad M = \left[\frac{\lambda}{b} - \left(\frac{p}{2\pi}\right)^2\left(\frac{\lambda}{b}\right)^3\right]^{-1/2}$$

From this equation it is clear that for a constant value of λ/b the attenuation decreases rapidly as b increases. For low attenuation it is then advisable to use as large a tube as permissible. However, the attenuation also varies with (λ/b), and the optimum value of (λ/b) is found by minimizing M. Let $(\lambda/b) = h$, and $(p/2\pi) = q$.

$$M = (h - q^2h^3)^{-1/2}$$

$$\frac{dM}{dh} = -\tfrac{1}{2}(h - q^2h^3)^{-3/2}(1 - 3q^2h^2) = 0$$

or

$$1 - 3q^2h^2 = 0$$

from which

$$h = \frac{\lambda}{b} = \frac{2\pi}{\sqrt{3}\,p} = 1.51$$

Actually the value of h is not very critical, and values between 1.0 and 2.0 are satisfactory. Thus the criteria for low attenuation are to set (1) b as large as possible and (2) (λ/b) at about 1.5. However, if the frequency to be transmitted is fixed, then the value of b is controlled by the factor 1.5.

128. Comparison of Wave Guide with Coaxial Cable. As an illustrative example, let it be required to transmit a frequency of 2000 megacycles per second and to compare the attenuation when using a wave guide with that obtained when using a coaxial line of the same diameter whose ratio of radii has the optimum value of 3.6. (See Art. 41.) The relation $\lambda/b = 1.5$ for the $TM_{0,1}$ mode gives the radius of the cylindrical guide as

$$b = \frac{\lambda}{1.5} = \frac{c}{1.5f} = \frac{3 \times 10^8}{1.5 \times 2000 \times 10^6} = 0.10 \text{ meter}$$

The cut-off frequency for this tube is

$$\begin{aligned} f_0 = \frac{cp}{2\pi b} = \frac{3 \times 10^8 \times 2.405}{2\pi \times 0.10} &= 1148 \times 10^6 \text{ cycles/sec} \\ &= 1148 \text{ megacycles/sec} \end{aligned}$$

The attenuation is given by equation 14–45 in which

$$\begin{aligned} f &= 2000 \times 10^6 \text{ cycles/sec} \\ \varepsilon = \epsilon_0 &= 1/(36\pi \times 10^9) \text{ farad/meter} \\ \mu_0 &= 4\pi \times 10^{-7} \text{ henry/meter} \\ g &= 6 \times 10^7 \text{ mho-m/sq m} \\ b &= 0.10 \text{ meter} \end{aligned}$$

On substitution of these quantities, α becomes

$$\begin{aligned} \alpha &= 3.72 \times 10^{-4} \text{ neper/meter} \\ &= 0.984 \text{ db/1000 ft} \end{aligned}$$

It will be assumed that equation 5–48 can be applied to the coaxial cable at 2000 megacycles per second. Changing to decibels per 1000 feet, equation 5–48 becomes

$$\alpha = 40.1 \times 10^{-6} \times \frac{\sqrt{f}\left(\frac{1}{a} + \frac{1}{b}\right)}{\log \frac{b}{a}} \text{ db/1000 ft} \qquad [5\text{–}48a]$$

Let the following values be used

$$\begin{aligned} f &= 2000 \times 10^6 \text{ cycles/sec} \\ a &= 2.78 \text{ cm (radius of inner conductor)} \\ b &= 10 \text{ cm (inner radius of outer conductor)} \end{aligned}$$

From equation 5–48a,

$$\alpha = \frac{40.1 \times 10^{-6}\sqrt{2000 \times 10^{6}}\left(\frac{1}{2.78} + \frac{1}{10}\right)}{\log 3.6}$$

$$= 1.48 \text{ db/1000 ft}$$

It is seen from the above that, in this illustration, the wave guide is to be preferred over the coaxial cable; however, the wave guide is used only for the very high frequencies because of the rather inconvenient sizes encountered at the low frequencies. Another advantage to add to the improvement in attenuation is the fact that the wave guide, in many cases, is the more easily constructed.

THE DOMINANT WAVE IN CYLINDRICAL GUIDES

129. The Dominant Wave in Cylindrical Guides. The dominant or $TE_{0,1}$ wave for rectangular guides was considered in Chapter XIII.

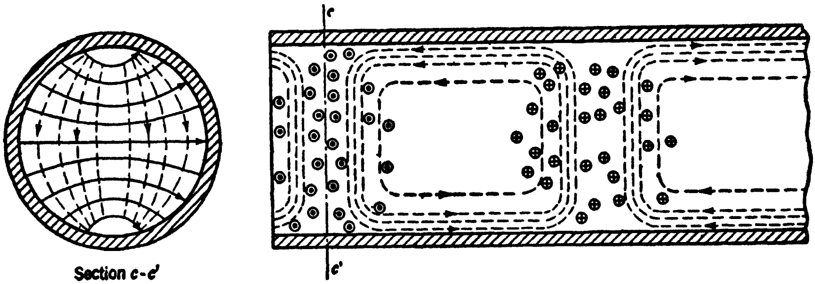

FIG. 14–8. Field configuration in cylindrical guides, $TE_{1,1}$ type.
——— Electric lines of force.
- - - - - Magnetic lines of force.
The circles with crosses and dots represent direction of electric lines according to the usual convention.

From the standpoint of logical sequence it might be inferred that this same type of wave, since it may exist in a cylindrical guide, should have been considered first in the present chapter. However, since the mathematical development for the dominant wave in a cylindrical guide (designated $TE_{1,1}$) is somewhat more involved than that for the "longitudinal" or $TM_{0,1}$ wave, the treatment of the dominant wave was delayed until more familiarity with the mathematics had been gained. The mode of vibration now considered is shown in Fig. 14–8, and the similarity to the corresponding mode for rectangular guides shown in Figs. 12–2a and 12–7 is evident.

The dominant mode of vibration may be obtained in a manner similar to that employed for the rectangular guide, that is, by exciting by means of a transverse rod as shown in Fig. 14–9. An examination of Fig. 14–8 shows that there is no component of the electric field parallel to the axis

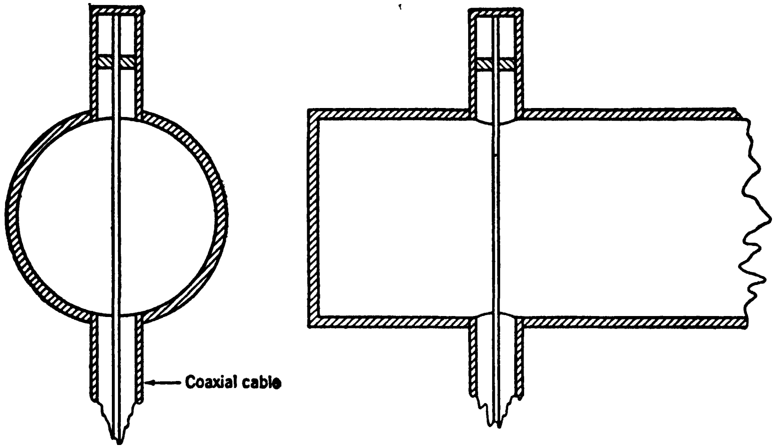

FIG. 14–9. Method of excitation for the $TE_{1,1}$ wave.

of the tube. The other components E_θ and E_r are present. All components of the magnetic field may be present. As evident from the foregoing treatment of the longitudinal wave, a certain amount of duplication between types of transmission is to be noted. However the development by the student of the basic equations for these three types of transmission in wave guides should provide a very good insight into the general problem.

130. The Differential Equation. The basis of the treatment is again of course equations A–67 to A–72 inclusive. The same assumptions as to variations of E and H with t and z are to be made as previously employed. The equations can be written as follows after these simplifications have been applied, remembering that $E_z = 0$:

$$\frac{\partial \mathcal{H}_z}{r\partial\theta} + \gamma\mathcal{H}_\theta = (g + j\omega\varepsilon)\mathcal{E}_r \qquad [14\text{–}52]$$

$$-\gamma\mathcal{H}_r - \frac{\partial \mathcal{H}_z}{\partial r} = (g + j\omega\varepsilon)\mathcal{E}_\theta \qquad [14\text{–}53]$$

$$\frac{1}{r}\left(\frac{\partial r\mathcal{H}_\theta}{\partial r} - \frac{\partial \mathcal{H}_r}{\partial\theta}\right) = 0 \qquad [14\text{–}54]$$

$$\gamma\mathcal{E}_\theta = -j\omega\mu_0\mathcal{H}_r \qquad [14\text{–}55]$$

$$-\gamma \mathcal{E}_r = -j\omega\mu_0 \mathcal{H}_\theta \qquad [14\text{–}56]$$

$$\frac{1}{r}\left(\frac{\partial r \mathcal{E}_\theta}{\partial r} - \frac{\partial \mathcal{E}_r}{\partial \theta}\right) = -j\omega\mu_0 \mathcal{H}_z \qquad [14\text{–}57]$$

From equation 14–55,

$$\gamma \frac{\partial \mathcal{E}_\theta}{\partial \theta} = -j\omega\mu_0 \frac{\partial \mathcal{H}_r}{\partial \theta}$$

Multiplying equation 14–56 by r and taking the derivative with respect to r gives

$$\gamma \frac{\partial r \mathcal{E}_r}{\partial r} = j\omega\mu_0 \frac{\partial r \mathcal{H}_\theta}{\partial r}$$

From these two equations, using equation 14–54

$$\frac{\partial \mathcal{E}_\theta}{\partial \theta} = -\frac{\partial r \mathcal{E}_r}{\partial r} \qquad [14\text{–}58]$$

and

$$\frac{\partial^2 \mathcal{E}_\theta}{\partial r \partial \theta} = -\frac{\partial^2 r \mathcal{E}_r}{\partial r^2} \qquad [14\text{–}59]$$

Differentiate equation 14–57 with respect to θ, divide by r, and rearrange.

$$\frac{j}{{}^2\omega\mu_0}\left(\frac{\partial^2 r \mathcal{E}_\theta}{\partial r \partial \theta} - \frac{\partial^2 \mathcal{E}_r}{\partial \theta^2}\right) = \frac{1}{r} \cdot \frac{\partial \mathcal{H}_z}{\partial \theta}$$

Substituting this expression and that of equation 14–56 into equation 14–52

$$\frac{j}{r^2\omega\mu_0}\left(\frac{\partial^2 r \mathcal{E}_\theta}{\partial r \partial \theta} - \frac{\partial^2 \mathcal{E}_r}{\partial \theta^2}\right) - \frac{j\gamma^2}{\omega\mu_0} \mathcal{E}_r = (g + j\omega\varepsilon)\mathcal{E}_r \qquad [14\text{–}60]$$

Rearrangement of this equation will give

$$-\frac{1}{r^2}\left(r \frac{\partial^2 \mathcal{E}_\theta}{\partial r \partial \theta} + \frac{\partial \mathcal{E}_\theta}{\partial \theta} - \frac{\partial^2 \mathcal{E}_r}{\partial \theta^2}\right) = (jg\omega\mu_0 - \omega^2\mu_0\varepsilon - \gamma^2)\mathcal{E}_r$$

Using equations 14–58 and 14–59 this becomes

$$r \frac{\partial^2 r \mathcal{E}_r}{\partial r^2} + \frac{\partial r \mathcal{E}_r}{\partial r} + \frac{\partial^2 \mathcal{E}_r}{\partial \theta^2} = -h^2 r^2 \mathcal{E}_r \qquad [14\text{–}61]$$

where

$$h^2 = \omega^2\mu_0\varepsilon + \gamma^2 - jg\omega\mu_0$$

131. Solution of the Differential Equation. Multiply equation 14–61 through by r, and consider that the new function to be found is $r\mathcal{E}_r \equiv Q$.

$$r^2 \frac{\partial^2 Q}{\partial r^2} + r \frac{\partial Q}{\partial r} + \frac{\partial^2 Q}{\partial \theta^2} = -h^2 r^2 Q \qquad [14\text{–}62]$$

Let Q be represented by RT, where R is the portion of the solution dependent only on r and T is the part dependent only on θ. Then

$$r^2T\frac{\partial^2 R}{\partial r^2} + rT\frac{\partial R}{\partial r} + R\frac{\partial^2 T}{\partial \theta^2} = -h^2r^2RT$$

or

$$\frac{r^2}{R}\cdot\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\cdot\frac{\partial R}{\partial r} + \frac{1}{T}\cdot\frac{\partial^2 T}{\partial \theta^2} = -h^2r^2 \qquad [14\text{–}63]$$

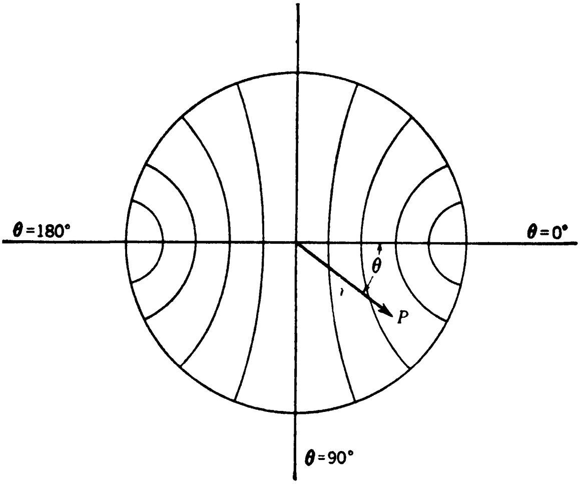

FIG. 14–10. Coordinate system for the $TE_{1,1}$ wave in cylindrical wave guides. The z axis projects into paper.

Since the third term depends only on θ, it must be independent of the rest of the equation. Therefore it is a constant with respect to the r terms and can be represented by a constant C_1. Then

$$\frac{\partial^2 T}{\partial \theta^2} = -C_1T \qquad [14\text{–}64]$$

and

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial R}{\partial r} + \left(h^2 - \frac{C_1}{r^2}\right)R = 0 \qquad [14\text{–}65]$$

The solution of equation 14–64 is

$$T = C_2 \sin\sqrt{C_1}\,\theta + C_3\cos\sqrt{C_1}\,\theta \qquad [14\text{–}66]$$

where T is the part of Q $(= r\mathfrak{E}_r)$ dependent on θ. From Fig. 14–10 it is seen that there are two values of θ where $r\mathfrak{E}_r$ will be zero, 0° and 180°.

Thus two boundary conditions exist for equation 14–66.

$$0 = C_2 \sin \sqrt{C_1}\, 0 + C_3 \cos \sqrt{C_1}\, 0$$

$$0 = C_2 \sin \sqrt{C_1}\, \pi + C_3 \cos \sqrt{C_1}\, \pi$$

From these equations it is seen that $C_3 = 0$ and that $\sin \sqrt{C_1}\, \pi = 0$. Thus the simplest condition results if $\sqrt{C_1} = 1$, and $C_1 = 1$ satisfies the condition. Thus

$$T = C_2 \sin \theta \qquad [14\text{–}67]$$

Equation 14–65 now becomes

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \left(h^2 - \frac{1}{r^2}\right) R = 0 \qquad [14\text{–}68]$$

The solution of this equation is known to be[6]

$$R = FJ_1(hr) + GY_1(hr)$$

The fact that $Y_1(hr)$ becomes infinite at $r = 0$ causes it to be unsuitable

[6] In Appendix VII it is shown that $J_1(r)$ and $Y_1(r)$ are solutions of

$$\frac{d^2y}{dr^2} + \frac{1}{r}\cdot\frac{dy}{dr} + \left(1 - \frac{1}{r^2}\right) y = 0 \qquad [A]$$

It will be shown that, if r in the above equation is changed to hz, then $J_1(hz)$ and $Y_1(hz)$ will be solutions of

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(h^2 - \frac{1}{z^2}\right) y = 0 \qquad [B]$$

Substituting hz for r into equation A

$$\frac{dy}{dr} = \frac{1}{h}\frac{dy}{dz}$$

$$\frac{d^2y}{dr^2} = \frac{d}{dr}\frac{dy}{dr} = \frac{d}{hdz}\left(\frac{1}{h}\frac{dy}{dz}\right) = \frac{1}{h^2}\frac{d^2y}{dz^2}$$

Therefore equation A becomes

$$\frac{1}{h^2}\cdot\frac{d^2y}{dz^2} + \frac{1}{hz}\cdot\frac{1}{h}\cdot\frac{dy}{dz} + \left(1 - \frac{1}{h^2z^2}\right) y = 0$$

Multiplying through by h^2

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(h^2 - \frac{1}{z^2}\right) y = 0 \qquad [C]$$

Since the solution of equation A is $CJ_1(r) + DY_1(r)$, then the solution of equation C must be $C'J_1(hz) + D'Y_1(hz)$.

for use as part of this solution, so the complete solution may be written

$$Q = r\mathcal{E}_r = RT = (C_2 \sin\theta)FJ_1(hr)$$

$$= AJ_1(hr)\sin\theta \qquad [14\text{–}69]$$

where A is a constant depending on the excitation. Thus

$$\mathcal{E}_r = \frac{A}{r} J_1(hr)\sin\theta \qquad [14\text{–}70]$$

It is now necessary to determine $\mathcal{E}_\theta$ in order to apply the boundary condition that $\mathcal{E}_\theta = 0$ when $r = b$. Proceed as follows:

$$\frac{\partial r\mathcal{E}_r}{\partial r} = AhJ_1'(hr)\sin\theta \qquad [14\text{–}71]$$

and, from equation 14–58,

$$\frac{\partial \mathcal{E}_\theta}{\partial\theta} = -AhJ_1'(hr)\sin\theta$$

$$\mathcal{E}_\theta = -AhJ_1'(hr)\int \sin\theta d\theta$$

$$= AhJ_1'(hr)\cos\theta + B$$

The boundary conditions for $\mathcal{E}_\theta$ are

$$\mathcal{E}_\theta = 0 \text{ at } r = b \quad \text{(radius of the tube)}$$

$$\mathcal{E}_\theta = 0 \text{ at } \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \quad (\text{any } r)$$

Accordingly

$$0 = AhJ_1'(hr)\cos\frac{\pi}{2} + B$$

and, since $\cos(\pi/2) = 0$, $B = 0$. Then

$$\mathcal{E}_\theta = AhJ_1'(hr)\cos\theta \qquad [14\text{–}72]$$

and, at $r = b$,

$$0 = AhJ_1'(hb)\cos\theta$$

Therefore

$$J_1'(hb) = 0 \qquad [14\text{–}73]$$

This equation[7] is satisfied by placing $hb = y = 1.84$. It is thus found that

$$hb = y = 1.84 = b\sqrt{\omega^2\mu_0\varepsilon + \gamma^2 - jg\omega\mu_0} \qquad [14\text{–}74]$$

However, for the interior of the tube, $g = 0$, so that h will be used to represent

$$h = \sqrt{\omega^2\mu_0\varepsilon + \gamma^2} = \frac{y}{b}$$

and

$$\gamma^2 = \left(\frac{y}{b}\right)^2 - \omega^2\mu_0\varepsilon$$

or

$$= \alpha + j\beta = \sqrt{\left(\frac{y}{b}\right)^2 - \omega^2\mu_0\varepsilon}$$

Again the only type of transmission which is of interest is that which is carried on with no attenuation; so

$$= \sqrt{\omega^2\mu_0\varepsilon - \left(\frac{y}{b}\right)^2}$$

$$= \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{y}{b}\right)^2} \qquad [14\text{–}75]$$

132. Components of the Field. Two of the field components have already been found. They are

$$\mathcal{E}_r = \frac{A}{r} J_1(hr) \sin\theta \qquad [14\text{–}70]$$

$$\mathcal{E}_\theta = AhJ_1'(hr) \cos\theta \qquad [14\text{–}72]$$

Also

$$\mathcal{E}_z = 0$$

[7] McLachlan, N. W., *Bessel Functions for Engineers*, New York, Oxford University Press, 1934. $J_1'(hb)$ represents the derivatives of $J_1(hb)$ with respect to hb. The expression for this derivative is

$$hbJ_1'(hb) = J_1(hb) - hbJ_2(hb)$$

from which

$$J_1'(hb) = \frac{J_1(hb)}{hb} - J_2(hb)$$

Through the use of tables of J_1 and J_2 this equation can be solved by trial with the condition that

$$\frac{J_1(hb)}{hb} = J_2(hb)$$

since $J_1'(hb) = 0$ from equation 14–73. It is found by this means that $hb = 1.84$.

Two components of the H field can be found from equations 14–55 and 14–56. From equation 14–55,

$$\mathcal{H}_r = \frac{j\gamma}{\omega\mu_0}\,\mathcal{E}_\theta$$

$$= \frac{-\beta A h}{\omega\mu_0}\,J_1'(hr)\cos\theta \tag{14–76}$$

From equation 14–56,

$$\mathcal{H}_\theta = \frac{-j\gamma}{\omega\mu_0}\,\mathcal{E}_r$$

$$= \frac{\beta A}{\omega\mu_0 r}\,J_1(hr)\sin\theta \tag{14–77}$$

From equation 14–53,

$$\frac{\partial\mathcal{H}_z}{\partial r} = -\gamma\mathcal{H}_r - j\omega\varepsilon\mathcal{E}_\theta \qquad (g = 0)$$

$$= \frac{j\beta^2 A h}{\omega\mu_0}\,J_1'(hr)\cos\theta - j\omega\varepsilon A h J_1'(hr)\cos\theta$$

$$= -jAh\left(\frac{-\beta^2}{\omega\mu_0} + \omega\varepsilon\right)J_1'(hr)\cos\theta \tag{14–78}$$

and

$$\frac{\partial\mathcal{H}_z}{\partial r} = -jAh\left(\frac{h^2}{\omega\mu_0}\right)J_1'(hr)\cos\theta$$

from which

$$\mathcal{H}_z = \frac{-jAh^2}{\omega\mu_0}\,J_1(hr)\cos\theta \tag{14–79}$$

133. Properties of the Transmission. Since β for this mode of transmission differs from the β in previous modes only through the quantity y, the following expressions can be written immediately:

$$\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{y}{b}\right)^2} \quad \text{radians/meter} \tag{14–75}$$

where mks values are used for c and b,

$$f_0 = \frac{cy}{2\pi b} \quad \text{cycles/sec} \tag{14–80}$$

$$v_p = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{y}{b}\right)^2}} = \frac{c}{\sqrt{1 - \left(\frac{\lambda_a}{3.41b}\right)^2}} \text{ meters/sec} \quad [14\text{–}81]$$

$$\lambda_g = \frac{2\pi}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{y}{b}\right)^2}} \text{ meters} \quad [14\text{–}82]$$

Also it can be shown that

$$Z_0 = \frac{353}{\sqrt{1 - \left(\frac{\lambda_a}{3.41b}\right)^2}} = \frac{353}{\sqrt{1 - \left(\frac{f_0}{f}\right)^2}} \text{ ohms} \quad [14\text{–}83]$$

The curve of attenuation versus frequency for this mode of vibration is quite similar to that of the $TM_{0,1}$ mode, both curves having a minimum.

In this connection it is interesting to note that the attenuation of one of the possible modes in cylindrical guides, the $TE_{0,1}$, decreases continuously as the frequency is increased.

134. Comparison of the $TE_{1,1}$ Wave with the $TM_{0,1}$ Wave. Since the quantity y which appears in the expressions for the $TE_{1,1}$ wave is less than the p that appears in the characteristic equations of the $TM_{0,1}$ wave, it is clear that a given size of wave guide will transmit a much lower frequency in the $TE_{1,1}$ mode than in the other. In fact for a 3-inch tube the cut-off frequency for the $TM_{0,1}$ wave was found to be 3010 megacycles per second. For the $TE_{1,1}$ mode the cut-off frequency is, from equation 14–80,

$$f_0 = \frac{3 \times 10^8 \times 1.84}{2\pi \times 1.5 \times 0.0254} = 2310 \text{ megacycles/sec}$$

Also for a given tube size and a given frequency the phase velocity for the $TE_{1,1}$ or dominant mode will be less than for the $TM_{0,1}$ or longitudinal mode. This means of course that the group velocity will be greater for the $TE_{1,1}$ wave, and thus actually transmission will be speeded up.

PROBLEMS

14–1. For the type of wave considered in Art. 117 and for $z = t = 0$ plot the value of the E_z, E_r, and H_θ components from $r = 0$ to $r = b$. (Take $A = 1$, the tube diameter to be 3 inches, and $f = 4000$ megacycles/sec.) Repeat for $t = 0.626 \times 10^{-10}$ sec.

14–2. Find the values of E_z, E_r, and H_θ in Prob. 14–1 at a point given by $z = 5$ cm, $r = 1$ cm, and $t = 0$.

14-3. Given a cylindrical wave guide with a diameter of 4 inches. The wave to be transmitted is the type treated in Art. 117. Plot Z_0, v_p, v_g, and λ_g against frequency from f_0 to $3f_0$.

14-4. The tube in Prob. 14-3 is made of copper. Find the attenuation in db per mile at a frequency of 3500 megacycles/sec.

14-5. What is the optimum size of tube to transmit 3500 megacycles per second using a cylindrical copper guide, the excitation being of the type considered in Art. 117? What is the attenuation of the tube per mile in db?

14-6. A cylindrical guide has a diameter $d = 3$ in. and is transmitting the "dominant" wave considered in Art. 129, at a frequency 30 per cent above the cut-off frequency. What are the values of v_p, v_g, λ_g, and Z_0? Find the lowest diameter of tube which will transmit a dominant wave of 3000 megacycles per second.

14-7. Derive the numerical equations for the curves which represent the electric lines of force in a cylindrical wave guide having a radius of 5 centimeters when excited in the $TM_{0,1}$ mode. The curves are to pass through the following points:

(a) $z = 0,\ r = 0.5$ cm (c) $z = 0,\ r = 1.5$ cm

(b) $z = 0,\ r = 1.0$ cm (d) $z = 0,\ r = 2.0$ cm

(e) $z = 0,\ r = 2.5$ cm

The guide is transmitting at 3000 megacycles per second, and the time condition is $t = 0$.

14-8. Plot the equations derived in Prob. 14-7 for the electric lines of force.

CHAPTER XV

ELECTROMAGNETIC THEORY OF COAXIAL LINES

In the opening paragraph of Chapter XII mention was made of the fact that the theory of coaxial, or concentric, conductors can be handled on the basis of Maxwell's equations, although elementary calculations on such lines have been treated very briefly in earlier chapters by means of the ordinary line theory. It is now well to note that in transmission lines in general, as the frequency is indefinitely increased, the treatment by the electromagnetic theory is more and more justified. It is for this reason that, in recent years when greater emphasis is being placed on ultrahigh-frequency work, a knowledge of electromagnetic field theory is necessary to every student of communication engineering.

At first sight it may seem, after the treatment of the cylindrical wave guide in Chapter XIV, that very little change will be necessary in order to make the treatment applicable to the coaxial line. Such is the case. However, it will be noted that the simple act of placing a conductor in the center of a cylindrical guide, while it can still be taken care of by the same theory, will lead to somewhat different results in many respects. One of the great advantages of the electromagnetic theory is that it can be applied equally well to problems which are normally handled in quite different ways.

In this chapter only the elements of the theory of coaxial transmission are presented and the student is advised to refer to supplementary material available in published articles.[1]

135. The Differential Equation. Cylindrical coordinates fit naturally into the present problem. The aim is to set up a differential equation which will relate the various field components and provide a means for determining the characteristics of the propagation. This preliminary development for the coaxial line will parallel that for the cylindrical wave guide in all respects, and it is presented here again merely to prevent unnecessary confusion.

In Fig. 15-1 is shown a sectional view of such a line. The outer radius of the inner conductor is a, the inner radius of the outer con-

[1] "Electromagnetic Theory of Coaxial Transmission Lines and Cylindrical Shields," by S. A. Schelkunoff, *B.S.T.J.*, Oct., 1934.

ductor is b, and P represents a point within the enclosed space designated by the coordinates, r, θ, and z where z is taken as the direction of propagation. For such a transmission line, where circular symmetry exists, there is no variation with θ. Thus, in the set of equations A-67 to A-72, all derivatives with respect to θ will be zero. Also it will be

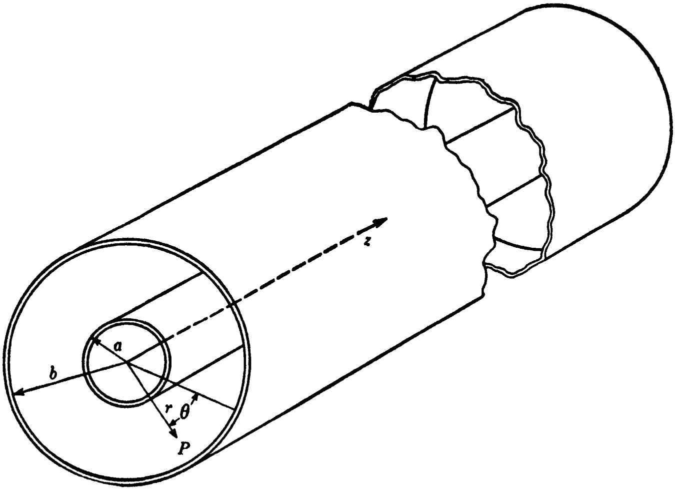

FIG. 15-1. Coaxial pair.

assumed that the time variation of all quantities is sinusoidal and given by $\epsilon^{j\omega t}$, so that in the equations $\mathcal{E}\epsilon^{j\omega t}$ and $\mathcal{H}\epsilon^{j\omega t}$ may be substituted for E and H. In the resulting equations the $\mathcal{E}$'s and $\mathcal{H}$'s will be functions of r and z only. Let it be further assumed that the material between the cylinders is a perfect dielectric, thereby making $g = 0$. The resulting equations can now be written by reference to equations A-67, A-68, A-69, A-70, A-71, and A-72.

$$-\frac{\partial \mathcal{H}_\theta}{\partial z} = j\omega\varepsilon\mathcal{E}_r \qquad [15\text{-}1]$$

$$\frac{\partial \mathcal{H}_r}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial r} = j\omega\varepsilon\mathcal{E}_\theta \qquad [15\text{-}2]$$

$$\frac{1}{r}\left(\frac{\partial r\mathcal{H}_\theta}{\partial r}\right) = j\omega\varepsilon\mathcal{E}_z \qquad [15\text{-}3]$$

$$-\frac{\partial \mathcal{E}_\theta}{\partial z} = -j\omega\mu_0\mathcal{H}_r \qquad [15\text{-}4]$$

$$\frac{\partial \mathcal{E}_r}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial r} = -j\omega\mu_0\mathcal{H}_\theta \qquad [15\text{–}5]$$

$$\frac{1}{r}\left(\frac{\partial r\mathcal{E}_\theta}{\partial r}\right) = -j\omega\mu_0\mathcal{H}_z \qquad [15\text{–}6]$$

It will be noted that the above equations fall into two groups, one involving $\mathcal{E}_\theta$, and one involving $\mathcal{H}_\theta$. On a line energized by connecting a generator to one end in the usual manner, E cannot have a θ component and therefore equations 15–2, 15–4, and 15–6 are eliminated. The remaining three equations will be further altered by the justified assumption, used in Chapter XIII, of making that part of each field component which depends upon z equal to

$$K\epsilon^{-\gamma z}$$

If this is done, all of the derivatives with respect to z can be eliminated in a manner similar to that used in Chapter XIII. From equation 15–1

$$\gamma\mathcal{H}_\theta = j\omega\varepsilon\mathcal{E}_r \qquad [15\text{–}7]$$

Expanding equation 15–3

$$\frac{r\partial\mathcal{H}_\theta}{\partial r} + \mathcal{H}_\theta = j\omega\varepsilon r\mathcal{E}_z \qquad [15\text{–}8]$$

From equation 15–5

$$\gamma\mathcal{E}_r + \frac{\partial \mathcal{E}_z}{\partial r} = j\omega\mu_0\mathcal{H}_\theta \qquad [15\text{–}9]$$

It will be noted that these equations are identical with equations 14–6, 14–7, and 14–5, respectively. The differential equation which expresses the relationship between $\mathcal{E}_z$ and r is given by equation 14–12 and is

$$\frac{\partial^2 \mathcal{E}_z}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial r} + k^2\mathcal{E}_z = 0 \qquad [15\text{–}10]$$

where

$$k^2 = \gamma^2 + \omega^2\mu_0\varepsilon$$

The solution is also

$$\mathcal{E}_z = AJ_0(kr) + BY_0(kr) \qquad [15\text{–}11]$$

where both J_0 and Y_0 are used, as shown in Appendix VII.

136. Evaluation of the Propagation Constant. The development for the coaxial pair now deviates from the course followed for the cylin-

drical wave guide. Since r ranges from a to b it never becomes either zero or infinite. For this reason the Y_0 part of the solution which is infinite at $r = 0$ can be retained. Along either conductor at $r = a$ and $r = b$, $\mathcal{E}_z$ must be zero because of the perfect conductivity of the material. Accordingly, two defining equations result. At $r = b$

$$\mathcal{E}_{zb} = AJ_0(kb) + BY_0(kb) = 0 \qquad [15\text{–}12]$$

and at $r = a$

$$\mathcal{E}_{za} = AJ_0(ka) + BY_0(ka) = 0 \qquad [15\text{–}13]$$

Let equations 15–12 and 15–13 be rearranged as follows:

$$-\frac{A}{B} = \frac{Y_0(kb)}{J_0(kb)}$$

and

$$-\frac{A}{B} = \frac{Y_0(ka)}{J_0(ka)}$$

or

$$-\frac{A}{B} = \frac{Y_0(kb)}{J_0(kb)} = \frac{Y_0(ka)}{J_0(ka)} \qquad [15\text{–}14]$$

In order to obtain approximate values of k the above equations can be rewritten in terms of circular functions.[2] For large x

$$J_0(x) \doteq \frac{\cos\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}} \qquad [15\text{–}15]$$

$$Y_0(x) \doteq \frac{\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}} \qquad [15\text{–}16]$$

Using these two approximations, equation 15–14 can be written

$$-\frac{A}{B} = \frac{\sin\left(kb - \frac{\pi}{4}\right)}{\cos\left(kb - \frac{\pi}{4}\right)} = \frac{\sin\left(ka - \frac{\pi}{4}\right)}{\cos\left(ka - \frac{\pi}{4}\right)} \qquad [15\text{–}17]$$

[2] Gray, A., and G. B. Mathews, *A Treatise on Bessel Functions*, New York, The Macmillan Co., 1895. Kármán, T., and M. A. Biot, *Mathematical Methods in Engineering*, p. 53, New York, McGraw-Hill Book Co., 1940.

This equation will be satisfied if

$$n\pi + ka - \frac{\pi}{4} = kb - \frac{\pi}{4}$$

or

$$k(b - a) = n\pi$$

from which

$$k = \frac{n\pi}{b - a} \qquad [15\text{–}18]$$

However, k^2 has been given the role of $\gamma^2 + \omega^2\mu_0\varepsilon$, so

$$\gamma^2 = k^2 - \omega^2\mu_0\varepsilon$$

and

$$\gamma = \sqrt{\left(\frac{n\pi}{b - a}\right)^2 - \omega^2\mu_0\varepsilon} \qquad [15\text{–}19]$$

Assigning various values to n will give γ for the corresponding modes of transmission. It is seen that a cut-off frequency exists which, for air dielectric, is given by

$$\frac{\omega}{c} = \frac{n\pi}{b - a}$$

or

$$f_0 = \frac{nc}{2\,(b - a)}$$

The cut-off frequency, below which transmission will not take place, becomes zero for $n = 0$. This amounts to setting $k = 0$ and the mode obtained is treated in the following section.

137. Mode of Transmission for $k = 0$. It is obvious that equation 15–14 can be satisfied by making $k = 0$. This leads to the simplest possible mode of transmission and requires that E_z be zero everywhere, as will be shown subsequently. Let equation 15–7 be solved for $\mathcal{E}_r$ and the result substituted into equation 15–9.

$$\frac{\gamma^2\mathcal{H}_\theta}{j\omega\varepsilon} + \frac{\partial\mathcal{E}_z}{\partial r} = j\omega\mu_0\mathcal{H}_\theta$$

$$\mathcal{H}_\theta\frac{\gamma^2 + \omega^2\mu_0\varepsilon}{j\omega\varepsilon} = -\frac{\partial\mathcal{E}_z}{\partial r}$$

or

$$\frac{\partial\mathcal{E}_z}{\partial r} = -\frac{k^2}{j\omega\varepsilon}\mathcal{H}_\theta \qquad [15\text{–}20]$$

Since $k = 0$, $\partial \mathcal{E}_z/\partial r = 0$, and $\mathcal{E}_z$ is a constant as far as any change with respect to r is concerned. It was previously shown that $\mathcal{E}_z = 0$ at the boundaries. Thus $\mathcal{E}_z$ must be zero everywhere.

The fact that k has been chosen as zero requires further that

$$k^2 = \gamma^2 + \omega^2\mu_0\varepsilon = 0$$

or that

$$\gamma^2 = -\omega^2\mu_0\varepsilon$$

and

$$\gamma = j\omega\sqrt{\mu_0\varepsilon} \qquad [15\text{–}21]$$

The propagation constant is made up of two parts, an attenuation constant and a phase constant, that is

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu_0\varepsilon}$$

or

$$\beta = \omega\sqrt{\mu_0\varepsilon} \qquad [15\text{–}22]$$

However, the velocity of light, c, is equal to $1/\sqrt{\mu_0\epsilon_0}$; so the velocity of the wave transmitted by the line is

$$v_p = \frac{\omega}{\beta} = c \qquad [15\text{–}23]$$

138. Components of the Field. One component of the field has been shown to be zero everywhere under the assumption that $k = 0$. Equations 15–7, 15–8, and 15–9 may now be written

$$\gamma\mathcal{H}_\theta = j\omega\varepsilon\mathcal{E}_r \qquad [15\text{–}7]$$

From equation 15–8

$$\frac{\partial r\mathcal{H}_\theta}{\partial r} = 0 \qquad [15\text{–}24]$$

and from equation 15–9

$$\gamma\mathcal{E}_r = j\omega\mu_0\mathcal{H}_\theta \qquad [15\text{–}25]$$

Equations 15–7 and 15–25 can be shown to be consistent if $k = 0$. From equation 15–25, using equation 15–21

$$\mathcal{E}_r = \frac{j\omega\mu_0\mathcal{H}_\theta}{j\omega\sqrt{\mu_0\varepsilon}}$$

$$= \sqrt{\frac{\mu_0}{\varepsilon}}\,\mathcal{H}_\theta \qquad [15\text{–}26]$$

Equation 15–24 states that

$$\frac{\partial r\mathcal{H}_\theta}{\partial r} = 0$$

or

$$r\mathcal{H}_\theta = C$$

from which

$$\mathcal{H}_\theta = \frac{C}{r} \tag{15-27}$$

where C is a constant, independent of r. The usual expression for the field about a wire is $\mathcal{H}_\theta = 2I/r$ where I is in abamperes to give $\mathcal{H}_\theta$ in gilberts per centimeter. Since $\mathcal{H}_\theta$ is to be expressed in amperes per centimeter and I in amperes, the equation must be written

$$\mathcal{H}_\theta = \frac{2I}{4\pi r} = \frac{I}{2\pi r} \tag{15-28}$$

Thus the constant C in equation 15–27 is $I/2\pi$.

The field components now are

$$\mathcal{E}_r = \sqrt{\frac{\mu_0}{\varepsilon}}\mathcal{H}_\theta \tag{15-26}$$

and

$$\mathcal{H}_\theta = \frac{I}{2\pi r} \tag{15-28}$$

where the exponential term $\epsilon^{(-\gamma z + j\omega t)}$ has been omitted.

139. The Characteristic Impedance. It is a comparatively simple matter to find the voltage between the inner and outer conductors in terms of the current. Let equation 15–28 be substituted into equation 15–26.

$$\mathcal{E}_r = \sqrt{\frac{\mu_0}{\varepsilon}} \cdot \frac{I}{2\pi r} \tag{15-29}$$

The voltage between conductors is found by integrating $\mathcal{E}_r$ from the inner conductor surface to the outer conductor.

$$\begin{aligned} V &= \int_a^b \mathcal{E}_r dr = \sqrt{\frac{\mu_0}{\varepsilon}} \cdot \frac{I}{2\pi} \int_a^b \frac{dr}{r} \\ &= \frac{I}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon}} \ln \frac{b}{a} \end{aligned}$$

From this equation the ratio $V/I = Z_0$ is found to be

$$Z_0 = \frac{V}{I} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon}}\ln\frac{b}{a} \qquad [15\text{–}30]$$

By substituting the numerical values for μ_0 and ε ($= \epsilon_0$) there is obtained for air dielectric

$$Z_0 = 60\ln\frac{b}{a} \quad \text{ohms} \qquad [15\text{–}31]$$

$$= 138\log\frac{b}{a} \quad \text{ohms} \qquad [5\text{–}47]$$

140. Inductance and Capacitance of a Coaxial Line. As a concluding article it will be shown that the usual equations for inductance and capacitance of a coaxial line can be derived on the basis of the electromagnetic treatment. It has already been shown that $\mathcal{H}_\theta = I/2\pi r$ is consistent with the foregoing development and that E_z (or $\mathcal{E}_z$) is zero everywhere. Also, as has been done previously, that part of the field component which is dependent upon t will be omitted. Equation 15–5 can then be written

$$\frac{\partial \mathcal{E}_r}{\partial z} = \frac{-j\omega\mu_0 I}{2\pi r} \qquad [15\text{–}32]$$

and from equation 15–1

$$\frac{1}{2\pi r}\cdot\frac{\partial I}{\partial z} = -j\omega\varepsilon\mathcal{E}_r$$

or

$$\frac{1}{r}\cdot\frac{\partial I}{\partial z} = -j2\pi\omega\varepsilon\mathcal{E}_r \qquad [15\text{–}33]$$

Let these equations be integrated with respect to r from $r = a$ to $r = b$.

$$\frac{\partial}{\partial z}\int_a^b \mathcal{E}_r dr = \frac{-j\omega\mu_0 I}{2\pi}\int_a^b \frac{dr}{r}$$

or

$$\frac{\partial V}{\partial z} = \frac{-j\omega\mu_0 I}{2\pi}\ln\frac{b}{a} \qquad [15\text{–}34]$$

Also

$$\frac{\partial I}{\partial z}\int_a^b \frac{dr}{r} = -j2\pi\omega\varepsilon\int_a^b \mathcal{E}_r dr$$

$$\frac{\partial I}{\partial z}\ln\frac{b}{a} = -j2\pi\omega\varepsilon V$$

or

$$\frac{\partial I}{\partial z} = \frac{-j2\pi\omega\varepsilon V}{\ln \frac{b}{a}} \qquad [15\text{–}35]$$

where V is the potential difference between the elements of the coaxial line.

On the basis of ordinary transmission line theory the differential equations 15–34 and 15–35 may be written

$$\frac{\partial V}{\partial z} = -ZI \qquad [15\text{–}36]$$

and

$$\frac{\partial I}{\partial z} = -YV \qquad [15\text{–}37]$$

where a negative sign now appears in place of the positive sign in equations 5–1 and 5–2. The occurrence of negative signs in these equations is due to the fact that voltage and current drops are taken in the direction of propagation. The capital Z and Y are used in equations 15–36 and 15–37 to denote impedance and admittance per unit length, instead of lower case letters as in equations 5–1 and 5–2, because of the prior use of z as the coordinate axis in the direction of propagation. A comparison of equations 15–36 and 15–37 with equations 15–34 and 15–35 shows, on the basis of infinite conductivity of conductor and a perfect dielectric between conductors which is the basis of the present treatment, that

$$Z = \frac{j\omega\mu_0}{2\pi} \ln \frac{b}{a} = j\omega L \qquad [15\text{–}38]$$

and

$$Y = \frac{j2\pi\omega\varepsilon}{\ln \frac{b}{a}} = j\omega C \qquad [15\text{–}39]$$

where L and C are parameters of the line per unit length, measured in henrys and farads respectively.

From equations 15–38 and 15–39,

$$L = \frac{\mu_0}{2\pi} \ln \frac{b}{a} \quad \text{henry/unit length} \qquad [15\text{–}40]$$

$$C = \frac{2\pi\varepsilon}{\ln \frac{b}{a}} \quad \text{farad/unit length} \qquad [15\text{–}41]$$

In the rationalized mks system

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{henry/meter}$$

and

$$\epsilon_0 = 1/(36\pi \times 10^9) \quad \text{farad/meter}$$

On substitution

$$L = 2 \ln \frac{b}{a} \times 10^{-7} \quad \text{henry/meter} \qquad [1\text{–}24]$$

$$C = \frac{\epsilon_r}{18 \times 10^9 \ln \frac{b}{a}} \quad \text{farad/meter} \qquad [1\text{–}35]$$

Changing to henrys and microfarads per mile and to logarithm to the base 10,

$$L = 0.741 \log \frac{b}{a} \times 10^{-3} \quad \text{henry/mile} \qquad [1\text{–}25]$$

and

$$C = \frac{0.0388\epsilon_r}{\log \frac{b}{a}} \quad \mu\text{f/mile} \qquad [1\text{–}37]$$

These are the usual equations for L and C as defined and derived in Chapter I. It is seen that on the more general theory they occur as constants inherent to the system, depending only on the boundary conditions and the fundamental bases of Maxwell's equations.

This rather brief treatment of coaxial lines (and that in the previous chapters on wave guides) is to be looked on as merely introductory. It cannot be too greatly emphasized that the student should make himself thoroughly familiar with the electromagnetic treatment of electrical-engineering problems and with the mathematical methods which are basic to this treatment.

CHAPTER XVI

TRANSMISSION-LINE EXPERIMENTS

The aim of the present chapter is to outline briefly a number of experiments which are designed to lend considerable aid to the better understanding of various portions of the text material. Detailed procedures will not be given because of the fact that the equipment available in different communication laboratories varies greatly. It is assumed, however, that the more usual pieces of test equipment are at hand, such as the impedance bridge, vacuum-tube voltmeters, and oscillators. An artificial transmission line will be necessary for the performance of several experiments. If such a line is not already available, one can be constructed with little difficulty by calculating equivalent T sections by the method given in Chapter VII. The effective length of each T section should be approximately 1/15 of the total length of the line, and the line should be sufficiently long to subtend slightly more than one wavelength at the usual test frequency, which for convenience may be taken to be 796 cycles per second.

The experiments on wave guides involve equipment not usually available. However, these experiments have been included because of the increasing importance of high-frequency technique, and many laboratories may find it advantageous to acquire the necessary equipment. These two experiments should offer suggestions concerning the types of apparatus needed for elementary work.

1. Determination of Equivalent T and π Sections.

Reference: Chapter II.

Apparatus needed: Several three- and/or four-terminal boxes containing " unknown " T or π sections.

Impedance bridge.

Oscillator (voice frequencies).

Exercises: Label the input terminals of the unknown networks a and b and the output terminals c and d. Measure Z_{abo}, Z_{abs}, Z_{cdo}, and Z_{cds} by means of the impedance bridge at a frequency of 796 cycles per second for each network. Record the data as: R_{abo}, L_{abo}, (C_{abo}), R_{abs}, L_{abs}, (C_{abs}), etc., and f. Repeat the measurements for a frequency of 1600 cycles per second.

Report: Calculate the elements of the equivalent T section of each network at both frequencies and draw a circuit diagram for each. Also calculate the elements of the π sections equivalent to the T sections at 796 cycles per second. Discuss fully. Are the elements of these sections constant or do they change with a change in frequency? Verify the validity of equation 2–31 by means of test data for each network and frequency employed.

2. Verification of Thévenin's Theorem and the Reciprocity Theorem.

Reference: Chapter III

Apparatus needed: Resistances.
Variable d-c voltage.
Ammeter and voltmeter

Exercises: (*a*) Thévenin's Theorem Set up a circuit such as shown in Fig. 16–1, preferably using resistances which are large compared to the internal resistance of the battery. Measure the value of the current I in the load resistance. Remove the resistance R_L and measure:

(1) E_{ab}.

(2) R_{ab} with E removed and the two terminals short-circuited together. It would be preferable to insert a resistance at E when the battery is removed of a value equal to the internal resistance of E.

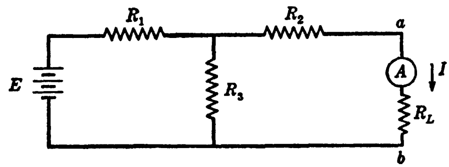

FIG 16–1 Test circuit for verification of Thévenin's theorem

Use these measured values to construct the circuit shown in Fig. 16–2, again using R_L. How does the load current compare with the value as measured before any change was made?

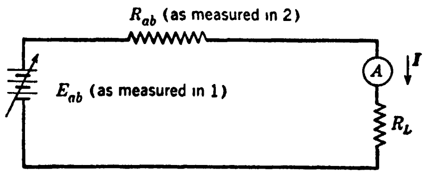

FIG 16–2 Test circuit constructed from data obtained in experiment 2(*a*).

Determine the values of E, R_1, R_2, R_3, and R_L, as used in Fig. 16–1. (Note that the resistance of the ammeter constitutes part of R_L.) Determine by calculation the values of R_{ab} and E_{ab} and compare with the values used in the circuit of Fig. 16–2.

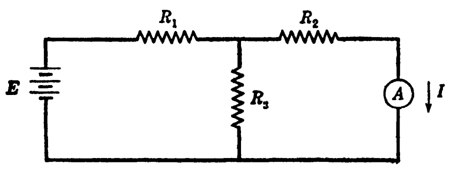

FIG 16–3 Test circuit for verification of reciprocity theorem

(b) Reciprocity Theorem. Set up a circuit such as shown in Fig. 16–3. Record the value of the current I. Interchange the source of voltage and the ammeter and again record the value of the current.

Report: Write a discussion of the results of this experiment and establish proof, by means of suitable equations, of the truth of these theorems.

3. Distribution of Voltage and Current along an Infinite Line.

References: Chapters IV and V.

Apparatus needed: Artificial line of approximately 15 sections.

Oscillator.

Vacuum tube voltmeter or other means of measuring either the absolute values of voltages and currents or the ratios of voltages and currents at any point along the line to input voltages and currents.

Exercises: Calculate the characteristic impedance, Z_0, for the test line at 796 cycles per second, using the known parameters of the line. (If these values are not known then Z_0 can be found by the methods of Experiment 7 and Chapter VII.) Terminate the line by means of an impedance equal to Z_0, thus causing the line to simulate one of infinite length. Make all of the necessary measurements, at 796 cycles per second, in order to obtain either V_1 and I_1, the voltage and current at the line input terminals, and V_n and I_n, the corresponding values at the end of the nth section, or the ratios V_n/V_1 and I_n/I_1.

Report: Plot the voltage and current ratios V_n/V_1 and I_n/I_1 against equivalent line length in miles. If the value of V_n and I_n were obtained, calculate the impedance at each junction between sections and plot Z against line length. The impedance at any point is $Z_n = V_n/I_n$. If the ratios V_n/V_1 and I_n/I_1 were obtained, then calculate the ratio of the impedance at any point to the input impedance as

$$\frac{\dfrac{V_n}{V_1}}{\dfrac{I_n}{I_1}} = \frac{\dfrac{V_n}{I_n}}{\dfrac{V_1}{I_1}} = \frac{Z_n}{Z_1}$$

and plot against line length. Determine α by means of data obtained from the above plotted curves, using the equations

$$\epsilon^{-\alpha l} = \frac{I_n}{I_1} \qquad\qquad \epsilon^{-\alpha l} = \frac{V_n}{V_1}$$

4. Distribution of Voltage and Current along Open- and Short-Circuit Lines.

Reference : Chapters V and VI.

Apparatus needed: Same as for Experiment 3.

Exercises: (*a*) With the output terminals of the line open-circuited, measure either the ratios V_n/V_1 and I_n/I_1 or the input voltage and current and the corresponding values, V_n and I_n, after each section of the artificial line, when using a frequency of 796 cycles per second.

(*b*) Repeat for a short-circuit line.

Report: Plot the voltage and current ratios V_n/V_1 and I_n/I_1 against equivalent line lengths in miles for both the open- and short-circuit lines. If the values of V_n and I_n were obtained, calculate the impedance at each junction between sections and plot curves of impedance against line length for both conditions of operation. If the ratios V_n/V_1 and I_n/I_1 were obtained, then calculate the ratio of the impedance at any point to the input impedance and plot against line length for both lines. Draw a curve through the intersections of the two voltage ratio curves and calculate the attenuation constant α from $\epsilon^{-\alpha l} = V_n/V_1$. Draw a curve through the intersections of the two current ratio curves and calculate α from $\epsilon^{-\alpha l} = I_n/I_1$. The wavelength λ can be obtained from the curves by noting the distances between appropriate minimum and maximum points on them. Using data obtained from the curves, calculate β, γ, λ, and v.

5. Determination of Velocity of Propagation.

References: Chapters V and VI.

Apparatus needed: Artificial line terminated in its Z_0.

Oscillator.

Cathode-ray oscilloscope.

Exercises: Connect the oscillator, set at 796 cycles per second, and one pair of oscilloscope plates across the input terminals of the line. With the amplitude of the input oscillation, as viewed on the oscilloscope, set at a convenient value, connect the other pair of plates across the ends of successive sections along the line. For each test make a note of the approximate phase relationship between input and output voltages, and, when sufficient sections have been placed in the line to obtain a phase shift of approximately 180° or some multiple thereof, adjust the frequency until the phase shift is an exact multiple of 180°. Record the number of sections in the line, the phase rotation, and the frequency, *f*. Repeat for other frequencies considerably different from 796 cycles per second and from each other.

Report: For each test calculate the velocity of propagation and plot velocity against frequency. Calculate the propagation velocities for the frequencies used, from the known line parameters, and plot against frequency. Check against the value of v found from Experiment 4.

6. Open-End Voltage as a Function of Line Length.

Reference: Chapter VI.

Apparatus needed: Same as for Experiment 3.

Exercises: Using the known parameters of the line, determine by calculation whether Ferranti effect exists at 796 cycles per second and, if so, over what lengths of line. If there is no Ferranti effect at 796 cycles per second, determine a convenient frequency for which the effect will exist. Check the calculations by taking voltage readings at the open end of lines of lengths between the limits of which Ferranti effect should exist. A convenient way to make this check is to connect a cathode-ray oscilloscope first at the input terminals of the line and then at the end of the above lengths of open line, noting in each instance whether or not there is an appropriate increase in output voltage.

Measure, at a frequency for which Ferranti effect exists, the ratio of the output voltage of the open-circuit line to the input voltage at lengths equivalent to 1, 2, 3 $\cdots$ sections.

Report: Plot the ratio, V_{ro}/V_1, of open-end voltage to input voltage against equivalent line length in miles. Plot $\pm$ sinh a and sin b, as illustrated in Fig. 6–6, on the same sheet with the above ratio curve. Explain the significance of these curves. For what length of line is the open-end voltage a maximum? Under what conditions will the open-end voltage be larger than the sending-end value? What is the length or lengths of line for which the open-end voltage will be equal to the sending-end value? Check these values by proper interpretation of the test ratio curve and the plots of sinh a and sin b.

7. Determination of Characteristic Impedance.

Reference: Chapter VII.

Apparatus needed: Artificial line.

Impedance bridge.

Oscillator.

Exercises: Determine by means of the impedance bridge the necessary data for calculating open- and short-circuit impedances, Z_{so} and Z_{ss}, of the line over a frequency range of 500 to 2000 cycles per second, taking a set of measurements at 796 cycles per second. Record the data as: R_o, L_o, (C_o), R_s, L_s, (C_s), and f.

Report: Plot R_o, L_o, etc., against frequency. Calculate and plot Z_{so} and Z_{ss} against frequency. On the same sheet plot the angles of Z_{so} and Z_{ss} against frequency. Calculate Z_0 and its angle and plot both against frequency. Discuss these curves. Calculate α, β, γ, λ, v and R, L, G, and C for a frequency of 796 cycles per second, employing the method of Art. 59. Compare with corresponding items obtained from Experiments 3, 4, and 5. Explain how such measurements may be used to locate a fault (open circuit) on a transmission line.

8. Impedance Characteristics of Filters.

References: Chapters IX and X.

Apparatus needed: Constant-k, T-section, low- and high-pass filters having a characteristic impedance of approximately 600 ohms.

Terminating half-sections for one of the above filters.
Impedance bridge.
Oscillator.

Exercises: Terminate each filter with 600 ohms resistance and obtain the necessary data to calculate its input impedance over a frequency range which extends a considerable distance both above and below its respective cut-off frequency. Record the data as R_s, L_s, (C_s), and f.

Obtain the necessary data to calculate the open- and short-circuit impedances of the filters over the same frequency range and record: R_{so}, R_{ss}, L_{so}, (C_{so}), L_{ss}, (C_{ss}), and f. Connect the terminating half-sections to the appropriate filter and repeat the measurements for the determination of open- and short-circuit impedances. Again record the data as above.

Report: Plot input resistance, R_s, and input reactance, X_s, against frequency. Discuss the variation of these quantities with frequency and determine, by inspection of the curves, the approximate cut-off frequency of each filter. From the open- and short-circuit measurements determine the components R_0 and X_0 of the characteristic impedance, Z_0, of each filter as a function of frequency. Plot R_0 and X_0 against frequency and from these curves determine the cut-off frequencies. Compare these values of f_0 with the values obtained previously.

Calculate and plot the absolute value of Z_0 against f for the filter having the terminating half-sections when operating with and without the terminating half-sections. Discuss the differences which exist between the curves of R_0, X_0, and Z_0 for the two conditions of operating with and without the terminating half-sections.

9. Impedance Transformation.

Reference: Chapter XI.

Apparatus needed: Oscillator.

Transformers and reactance T networks.

Impedance bridge.

Exercises: (*a*) Terminate the transformers with the following loads and measure the input impedance at 796 cycles per second. (1) Resistance. (2) Capacitive reactance. (3) Inductive reactance. (4) Impedances involving combinations of the above.

(*b*) Measure, at 796 cycles per second, the input impedances of the reactive T sections when terminated in pure resistance loads which vary in value over a considerable range.

Report: List the input impedances measured in part *a* and compare with the terminating impedances. From the turn ratios of the transformers calculate the input impedance for each termination. Discuss any discrepancy which may appear.

Check the value of each input impedance obtained in part *b* by using the known elements of the T section and the theory of Chapter XI.

10. Impedance Matching by Short-Circuit Stubs.

References: Chapter XI.

Apparatus needed: Parallel-wire transmission line approximately three wavelengths long.

HF oscillator generating about 100 megacycles per second.

Voltage detector for determining voltages along the line.

Short-circuited stub approximately one-quarter wavelength long and having a Z_0 equal to that of the transmission line.

Terminating resistor, R.

Exercises: Set up the line to be energized from the oscillator and tune to resonance at the fundamental by means of a short-circuiting bar. Use a suitable detector to determine the resonant condition. By means of the detector and an additional short-circuiting bar or by moving the first shunting bar, find an adjacent bar position which will again produce resonance. The wavelength and hence the frequency are determined from these measurements.

Calculate the Z_0 of the line and the frequency of operation.

With the line open-circuited at the receiving end, determine, by means of a detector, points of voltage (or current) maxima. Using the detector, obtain data for plotting the approximate distribution of voltage (or current) along the line. Connect a resistance across the

receiver end of the line of a value different from the characteristic impedance of the line and again determine the distribution of voltage (or current). Note any change produced by the terminating resistor.

Calculate the length and position of a short-circuited stub which will match the line and cause reflection phenomena to disappear when the line is terminated by the above load resistance. Connect this matching stub to the energized line and again determine the voltage (or current) distribution. Make adjustments in the length of the stub and its position, noting in each instance whether any improvement occurs in the voltage (or current) distribution.

Report: Plot curves showing the voltage (or current) distribution along the line before and after matching. Explain the theory underlying this method. Check the position and length of the stub by use of Fig. 11–20, page 222.

11. Properties of Wave Guides.

References: Chapters XII, XIII, and XIV.

Apparatus needed: UHF generator (3000 megacycles).
Coaxial coupling line.
Two transfer sections (wave guide to coaxial line).
Horn.
Wire grid to fit opening of the horn.
Resonant chamber.
Pinch pipe.
Hand probe detector.

(NOTE: For 3000 megacycles per second the diameter of the guide should be about 3 inches. The opening of the horn should be about 12 inches.)

Exercises: (*a*) Set up equipment as shown in Fig. 16–4. Terminate the wave guide in a horn across the opening of which has been fastened a

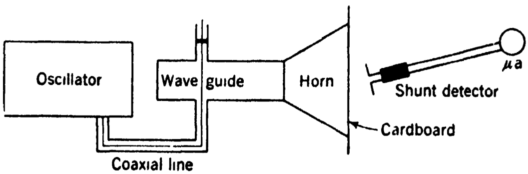

FIG. 16–4. Horn coupled to a wave guide.

cardboard panel. Using a hand probe, determine the pattern of the field. With a parallel-wire grid over the opening of the horn determine the effect on the field pattern as the grid is rotated.

(*b*) Replace the horn with a resonant chamber as shown in Fig. 16–5 and determine the positions of the plunger for obtaining two successive maxima or minima. Measure the wavelength λ_g.

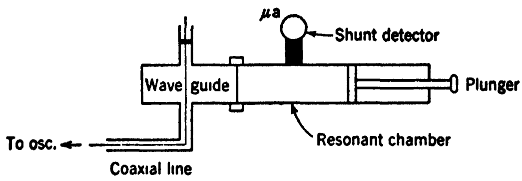

FIG. 16–5. Resonant chamber coupled to a wave guide.

(*c*) By means of a transfer section couple the wave guide to a coaxial wavemeter and measure the wavelength in air, λ_a. See Fig. 16–6.

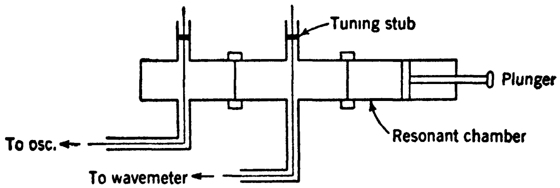

FIG. 16–6. Wavemeter coupled to a wave guide through a transfer section.

(*d*) Using a " pinch pipe " as shown in Fig. 16–7, determine the diameter of the tube for which cut-off occurs.

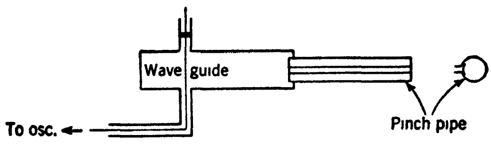

FIG. 16–7. The use of a pinch pipe in wave guides.

Report: Discuss the field patterns obtained in part *a* and compare with the theory as presented in Chapter XIV. Explain the effect of rotation of the wire grid in front of the horn. Check the value of λ_g by means of the equation

$$\lambda_g = \frac{\lambda_a \lambda_0}{\sqrt{\lambda_0^2 - \lambda_a^2}}$$

where $\lambda_0 = c/f_0$.

Check the diameter of the pinch pipe for which cut-off occurs with the relation $d_0 = 0.585\lambda_a$.

12. Properties of Wave Guides (*Continued*).

References: Chapters XII, XIII, and XIV.

Apparatus needed: The following apparatus is required in addition to that enumerated in Experiment 11.

Two rods, one of polystyrene *A* and one of polystyrene *B*, to fit resonant chamber.

Traveling detector.

Four irises as shown in Fig. 16–9.

Exercises: (*a*) Attach a resonant chamber with detector to a wave guide and source and insert an iris at the junction as shown in Fig. 16–8.

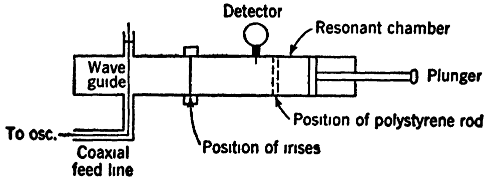

FIG. 16–8. Resonant chamber with iris and polystyrene rod.

Take readings of the detector current and plunger position over a range sufficient to obtain a good resonance curve. Repeat the test, using a different iris. Repeat, using the smallest iris and a rod of polystyrene *A* inserted in the resonant chamber between the detector and the plunger. Repeat, using a rod of polystyrene *B* together with the smallest iris.

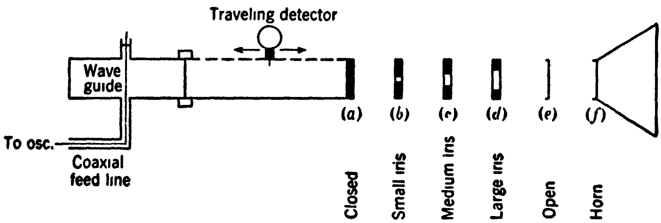

FIG. 16–9. Irises and traveling detector coupled to wave guide.

(*b*) Use the traveling detector in order to obtain data for plotting standing waves obtained when the detector section is terminated with a solid closure. Repeat for each of the other irises and also for an open end and the horn. See Fig. 16–9.

(*c*) Terminate the traveling detector section with a 90° bend in the horizontal plane, followed by a horn. Check this arrangement for reflections, which will be due to the presence of the bend. Repeat with the 90° bend in a vertical plane.

(*d*) With a transfer section attached to the traveling detector as shown in Fig. 16–10, feed energy into a properly terminated coaxial

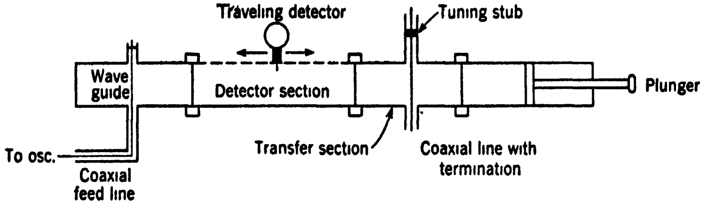

FIG. 16–10. Test for reflections from a transfer section.

cable. Check the presence of reflections under this condition of operation and with various adjustments.

Report: For part *a* plot current as a function of plunger position for each iris used. Discuss the relationship between these curves. Plot curves showing the standing waves of part *b* and discuss the effects of the various terminations used. Write a general discussion of the significance of the results of parts *c* and *d*.

APPENDIX I

AN INTRODUCTION TO FOURIER SERIES

In the Introduction it was stated that a transmission problem may be looked upon either as a transients problem or as one in which a number of frequencies are treated separately and the results combined. For instance, suppose that it is required to send a square-top pulse through a line. The conditions are that up to a time $t = 0$ the applied voltage is zero, from $t = 0$ to $t = t_1$ the voltage is E, and after $t = t_1$, E is zero again. It can be assumed that these pulses follow each other at regular intervals and that each interval is sufficiently long to allow the circuit to return to normal before a new pulse is applied. This is merely to prevent, for the moment, any complication due to overlapping. The problem, as outlined, is a transients one and could be solved as such.

Another way of approach is to consider that the series of pulses can be broken down into a large number of sinusoidal waves of different frequencies. When all the frequencies and corresponding amplitudes are known the student can treat each component as a separate problem, and when he has thus solved for all components he can find the final result by the principle of superposition (see Art. 21).

That such a series of square-top pulses can be reduced to a series of trigonometric functions has been shown to be the case. In fact, the process of reduction to a Fourier series, that is, to a series of trigonometric functions, can be applied to any periodically recurring phenomenon which, wholly or by parts, can be written as an equation. A graphical analysis can be applied, however, even when a part or the entire wave cannot be written in the form of an equation. The immediate problem is to find the coefficients of a trigonometric series such that the series will sum up to the required function. Let the function be $f(x)$.

Assume that $f(x)$ can be written as

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + a_n \sin nx$$
$$+ b_0 + b_1 \cos x + b_2 \cos 2x + \cdots + b_n \cos nx \qquad \text{[A-1]}$$

In this series it is necessary to find $a_1, a_2, a_3, \cdots a_n, b_0, b_1, b_2, b_3, \cdots b_n$. Obviously, in order to find these coefficients it is necessary to have as

many independent equations as there are coefficients. Multiply equation A–1 by $\sin mx$. Then there is obtained

$$f(x)\sin mx = a_n \sin nx \sin mx + b_0 \sin mx + b_n \cos nx \sin mx \qquad \text{[A–2]}$$

where n can have all values from 1 to infinity and the value of m can be given any arbitrary value. Let equation A–2, multiplied by dx, be integrated from 0 to 2π.

$$\int_0^{2\pi} f(x)\sin mx\,dx = a_n \int_0^{2\pi} \sin nx \sin mx\,dx + b_0 \int_0^{2\pi} \sin mx\,dx$$
$$+ b_n \int_0^{2\pi} \cos nx \sin mx\,dx \qquad \text{[A–3]}$$

Integrating these terms, the first one on the right becomes, after certain transformations

$$\frac{a_n}{2}\int_0^{2\pi} [\cos(nx - mx) - \cos(nx + mx)]\,dx = 0 \text{ for } n \neq m$$
$$= \frac{a_m}{2}\int_0^{2\pi} (1 - \cos 2mx)\,dx \text{ for } n = m$$
$$= \frac{a_m}{2}\int_0^{2\pi} dx - \frac{a_m}{4m}\int_0^{2\pi} \cos 2mx\,2m dx$$
$$= \frac{a_m}{2}(2\pi - 0) - \frac{a_m}{4m}(\sin 4m\pi - 0)$$
$$= a_m(\pi)$$

The second integral on the right is zero.

The third integral is

$$\frac{b_n}{2}\int_0^{2\pi} [\sin(n + m)x - \sin(n - m)x]\,dx$$

This integral will thus be equal to zero no matter whether $n = m$ or $\neq m$, for if $n = m$ the integral is

$$\frac{b_m}{2}\int_0^{2\pi} (\sin 2mx - 0)\,dx = 0$$

or if $n \neq m$ the integral is

$$\frac{b_n}{2}\int_0^{2\pi} (\sin px - \sin qx)\,dx = 0$$

where $p = (n + m)$ and $q = (n - m)$. Thus the following equation is obtained:

$$\int_0^{2\pi} f(x)\sin mx\,dx = a_m\pi$$

or solving for a_m

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx \qquad \text{[A-4]}$$

From this equation any one of the a's can be found.

To find the general b term, excluding b_0, multiply equation A–1 by $\cos mx\,dx$, whereupon the following equation is obtained:

$$f(x)\cos mx\,dx = a_n \sin nx \cos mx\,dx + b_0 \cos mx\,dx$$
$$+ b_n \cos nx \cos mx\,dx$$

Through a procedure similar to that used above the following equation results:

$$\int_0^{2\pi} f(x)\cos mx\,dx = b_m\pi$$

or

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\,dx \qquad \text{[A-5]}$$

In order to find the b_0 term, equation A–1 is multiplied by dx and integrated from 0 to 2π. Then

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} a_n \sin nx\,dx + \int_0^{2\pi} b_0\,dx + \int_0^{2\pi} b_n \cos nx\,dx \qquad \text{[A-6]}$$
$$= 0 + b_0(2\pi - 0) + 0$$
$$= 2\pi b_0$$

from which

$$b_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \qquad \text{[A-7]}$$

It is thus seen that all the a's and b's can be found from equations A–4, A–5, and A–7, and it would be advantageous if the three equations could be made to appear reasonable on a physical basis. Integration is a summation process, and, if equation A–1 is regarded merely as a rather complex wave and the net area between it and the x axis is to be found, it is seen that all of the sine and cosine terms are as often positive as they are negative and thus the net area associated with these terms becomes zero. The only remaining term is b_0, which, if present, is a constant of finite value either positive or negative and results in a definite area between the limits $x = 0$ and $x = 2\pi$. Since b_0 is a constant, the area

represented by the right-hand side of equation A–6 is merely $2\pi b_0$, which must equal the area $\int_0^{2\pi} f(x)\,dx$. Thus equation A–7 is obtained.

Consider now the physical aspects of determining a_m. Write

$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + b_0 + b_1 \cos x + b_2 \cos 2x + \cdots \tag{A–8}$$

If this equation is integrated directly as it stands it has been shown that equation A–7 is obtained and the a_n and b_n disappear. In order to keep a_n and b_n from disappearing it is necessary to multiply equation A–8 through by some factor which will retain one of the a terms or one of the b terms and cause all other terms including b_0 to fall out. This can only occur by multiplying by a function which will either cause the b_0 term to be zero or to be a function which has a negative area equal to its positive area. The simplest multiplier for the purpose is a trigonometric function, sine or cosine. Suppose that $\sin mx$ be assumed as the multiplier, then the right-hand side of equation A–8 will be made up of a number of terms of the form $\sin nx \sin mx$ and $\cos nx \sin mx$. Note that a cosine term multiplied by a sine term whether they are of the same frequency or not will always result in a periodic function which is positive as often as it is negative and symmetrical in area about the x axis. The area associated with the cosine terms in equation A–8 then is zero if taken over at least one cycle of this function. Since multiplying a cosine and sine function together always increases the frequency it is seen that the interval of integration, covering as it does one wavelength of the fundamental, must cover an integral number of wavelengths of any one of the terms. Thus the b_n terms drop out. The $\sin nx \sin mx$ terms will also be periodic, positive as often as negative — except for one term. When $m = n$, the term $\sin^2 mx$ obviously is always positive and thus will have a finite area lying between the curve and the x-axis. Since the $\sin^2 mx$ wave, having a maximum amplitude of a_m, is centered about an axis which is above the x axis by a distance $a_m/2$ the area under this curve from $x = 0$ to $x = 2\pi$ will be $(a_m/2)2\pi = a_m\pi$, which is an area equal to that given by $\int_0^{2\pi} f(x) \sin mx\,dx$. Thus equation A–4 is obtained.

A similar procedure will establish equation A–5.

Let these equations be applied to a square-top pulse in order to determine what frequencies must be employed to transmit such a pulse over a telephone line. Assume that the pulse is as shown in Fig. A–1. Here the conditions are: $f(x) = 1$, $x = 0$ to $x = \pi$; and $f(x) = 0$,

$x = \pi$ to $x = 2\pi$. The coefficients of the sine terms can be found from equation A–4.

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin mx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (1) \sin mx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (0) \sin mx\, dx$$

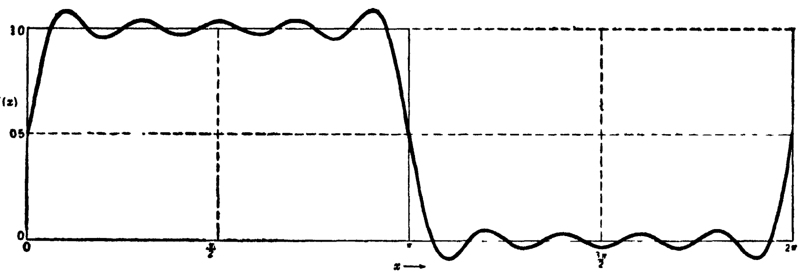

FIG. A–1. Plot of first five harmonics obtained in analysis of square-top wave.

This breaking up of the integral is justified by the fact that only the area under $f(x)$ multiplied by sin mx is being found and obviously the second part, from π to 2π, is zero. Then

$$a_m = \frac{1}{\pi}\int_0^{\pi} \sin mx\, dx$$

$$= \frac{1}{m\pi}(-\cos mx)\Big]_0^{\pi}$$

$$= \frac{1}{m\pi}(-\cos m\pi + \cos m0)$$

$$= \frac{1}{m\pi}(1 - \cos m\pi)$$

Whereupon

$$a_1 = \frac{1}{\pi}(1 - \cos \pi) = \frac{2}{\pi}$$

$$a_2 = \frac{1}{2\pi}(1 - \cos 2\pi) = 0$$

$$a_3 = \frac{1}{3\pi}(1 - \cos 3\pi) = \frac{2}{3\pi}$$

$$a_4 = 0$$

$$a_5 = \frac{2}{5\pi}$$

$$\cdots$$

The constant, b_0, can be found from equation A–7.

$$b_0 = \frac{1}{2\pi}\int_0^{\pi} (1)\, dx + \frac{1}{2\pi}\int_{\pi}^{2\pi} (0)\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}$$

Note here that b_0 turns out to be simply the average height of the wave.

The coefficients of the cosine terms can be found from equation A–5.

$$b_m = \frac{1}{\pi}\int_0^{\pi} (1) \cos mx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (0) \cos mx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos mx\, dx = \frac{1}{m\pi}(\sin mx)\Big]_0^{\pi}$$

$$= \frac{1}{m\pi}(\sin m\pi - \sin 0) = 0 \text{ for all values of } m$$

Thus the series of frequencies becomes

$$f(x) = b_0 + a_1 \sin x + a_3 \sin 3x + a_5 \sin 5x + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right] \qquad \text{[A–9]}$$

It is seen from this simple analysis that an infinite number of frequencies must be transmitted. They are all odd harmonics of the fundamental and their amplitudes slowly decrease with an increase in frequency. Eventually the amplitudes of the harmonics will become sufficiently low that they can be neglected. A direct-current term must also be transmitted. The presence of sharp corners on the original waveform is the cause of the large number of relatively high-amplitude harmonics which are necessary to represent this function correctly. This fact can be seen through inspection of Fig. A–1 where equation A–9 is plotted superimposed on the square-top pulse.

A comparison should be made between the number of terms needed to

reproduce the square-top pulse and the number needed for such a function as shown in Fig. A–2 where

$$f(x) = \sin x, \qquad x = 0 \text{ to } x = \pi$$

$$f(x) = 0, \qquad x = \pi \text{ to } x = 2\pi$$

The student should determine the harmonics necessary to represent this function in order to see that the amplitudes fall off more rapidly as the

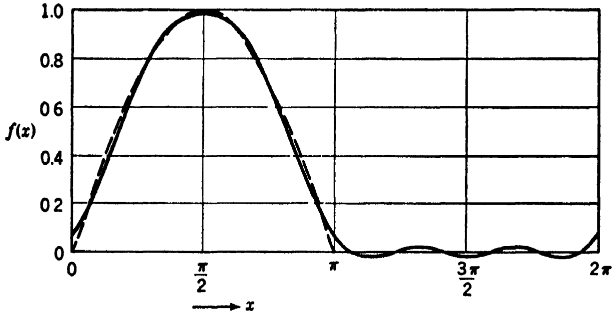

FIG. A–2. Plot of first four harmonics obtained in analysis of half-wave rectification curve.

frequency increases, than in the case of the square-top wave. It will be noted that the portions of poor fit are still at the sharp corners.

PROBLEMS

1. Plot equation A–9 for the terms shown. Let $0 < x < 2\pi$.

2. Find all the coefficients for terms up to the 5th harmonic in the wave represented by

$$0 < x < \pi \qquad f(x) = \sin x$$

$$\pi < x < 2\pi \qquad f(x) = 0$$

APPENDIX II

LOOP EQUATIONS

Occasions often arise when a straightforward and easily followed system of solving coupled networks is of great value. The use of "loop" equations will often facilitate such calculations. The determination of network currents by the use of loop equations and determinants will be illustrated by means of a number of typical examples. The basic principle of this method depends upon associating with each mesh, a loop current which is assumed to flow through all the component impedances appearing in the mesh under consideration.

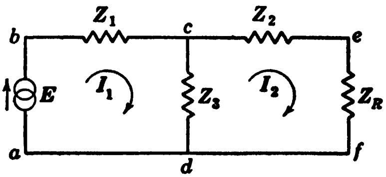

FIG. A–3. Network with two loop currents.

In Fig. A–3 is shown a network with two loop currents. In this network a current I_1, associated with the mesh *abcd,* flows through the generator E, the impedance Z_1, and the impedance Z_3. The current I_2, associated with the mesh *cefd,* flows through Z_2, Z_R, and Z_3. The fact is immediately noted that two currents appear to be flowing through Z_3, one in the direction *c–d* and the other in the direction *d–c.* The actual currents in the various impedances are as follows.

In Z_1 the current is I_1
In Z_2 " " " I_2
In Z_3 " " " $(I_1 - I_2)$ or $(I_2 - I_1)$
In Z_R " " " I_2

If the emf E is known there are thus only two unknowns, I_1 and I_2, and only two equations are needed to solve for them. The voltage rise in mesh *abcd* is E, and the drops are the drops through Z_1 and Z_3. Thus

$$E = Z_1I_1 + Z_3(I_1 - I_2)$$
$$0 = Z_2I_2 + Z_RI_2 + Z_3(I_2 - I_1)$$

and upon rearrangement these equations become

$$E = (Z_1 + Z_3)I_1 - Z_3I_2$$
$$0 = -Z_3I_1 + (Z_2 + Z_3 + Z_R)I_2$$

These equations can, of course, easily be solved for I_1 and I_2.

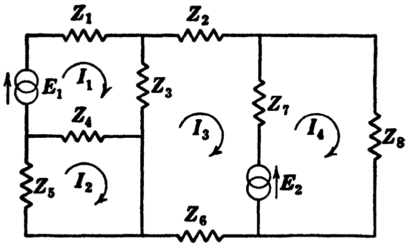

FIG. A–4. Network with four loop currents.

This method of solution will now be applied to a more complex network. Let the network be represented as in Fig. A–4. The loop equations are written as follows

$$E_1 = Z_1I_1 + Z_3(I_1 - I_3) + Z_4(I_1 - I_2)$$
$$0 = Z_5I_2 + Z_4(I_2 - I_1)$$
$$0 = Z_2I_3 + Z_7(I_3 - I_4) + E_2 + Z_6I_3 + Z_3(I_3 - I_1)$$

Notice that, in this last equation, the emf E_2 is written along with the impedance drops as a drop (a positive term) because in progressing around the loop one goes from the high potential side of E_2 to the low potential side.

$$E_2 = Z_7(I_4 - I_3) + Z_8I_4$$

These four equations may now be rewritten in better order with the emf's on the left side of the equality sign.

$$E_1 = (Z_1 + Z_3 + Z_4)I_1 - Z_4I_2 - Z_3I_3$$
$$0 = -Z_4I_1 + (Z_4 + Z_5)I_2$$
$$-E_2 = -Z_3I_1 + (Z_2 + Z_3 + Z_6 + Z_7)I_3 - Z_7I_4$$
$$E_2 = -Z_7I_3 + (Z_7 + Z_8)I_4$$

The determinants needed for the solution are

$$\Delta = \begin{vmatrix} (Z_1 + Z_3 + Z_4) & -Z_4 & -Z_3 & 0 \\ -Z_4 & (Z_4 + Z_5) & 0 & 0 \\ -Z_3 & 0 & (Z_2 + Z_3 + Z_6 + Z_7) & -Z_7 \\ 0 & 0 & -Z_7 & (Z_7 + Z_8) \end{vmatrix}$$

$$N_1 = \begin{vmatrix} E_1 & -Z_4 & -Z_3 & 0 \\ 0 & (Z_4 + Z_5) & 0 & 0 \\ -E_2 & 0 & (Z_2 + Z_3 + Z_6 + Z_7) & -Z_7 \\ E_2 & 0 & -Z_7 & (Z_7 + Z_8) \end{vmatrix}$$

$$N_2 = \begin{vmatrix} (Z_1 + Z_3 + Z_4) & E_1 & -Z_3 & 0 \\ -Z_4 & 0 & 0 & 0 \\ -Z_3 & -E_2 & (Z_2 + Z_3 + Z_6 + Z_7) & -Z_7 \\ 0 & E_2 & -Z_7 & (Z_7 + Z_8) \end{vmatrix}$$

$$N_3 = \begin{vmatrix} (Z_1 + Z_3 + Z_4) & -Z_4 & E_1 & 0 \\ -Z_4 & (Z_4 + Z_5) & 0 & 0 \\ -Z_3 & 0 & -E_2 & -Z_7 \\ 0 & 0 & E_2 & (Z_7 + Z_8) \end{vmatrix}$$

$$N_4 = \begin{vmatrix} (Z_1 + Z_3 + Z_4) & -Z_4 & -Z_3 & E_1 \\ -Z_4 & (Z_4 + Z_5) & 0 & 0 \\ -Z_3 & 0 & (Z_2 + Z_3 + Z_6 + Z_7) & -E_2 \\ 0 & 0 & -Z_7 & E_2 \end{vmatrix}$$

The loop currents are given by the following equations:

$$I_1 = \frac{N_1}{\Delta} \qquad I_3 = \frac{N_3}{\Delta}$$

$$I_2 = \frac{N_2}{\Delta} \qquad I_4 = \frac{N_4}{\Delta}$$

After the loop currents have been determined, the individual impedance currents are easily found by the following relations:

$$\text{Current in } Z_3 = I_1 - I_3$$
$$\text{Current in } Z_4 = I_1 - I_2$$
$$\text{Current in } Z_7 = I_3 - I_4$$

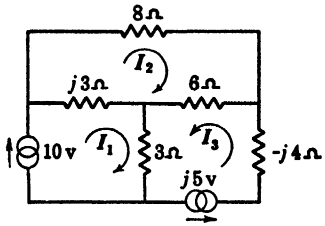

FIG. A-5. Network with three loop currents.

As a further illustration of the method, a three-mesh network with known impedances and emf's will be solved. Let this circuit be represented in Fig. A-5. The direction in which the arrows are drawn is immaterial so long as consistency is employed in writing the equation. In the network

$$10 = j3(I_1 - I_2) + 3(I_1 + I_3)$$

Note that for the 3-ohm resistance the total current is $(I_1 + I_3)$ because of the manner in which the arrow directions were taken.

$$0 = 8I_2 + 6(I_2 + I_3) + j3(I_2 - I_1)$$
$$j5 = -j4I_3 + 6(I_3 + I_2) + 3(I_3 + I_1)$$

These equations are written:

$$10 = (3 + j3)I_1 - j3I_2 + 3I_3$$
$$0 = -j3I_1 + (14 + j3)I_2 + 6I_3$$
$$j5 = 3I_1 + 6I_2 + (9 - j4)I_3$$

The determinants are:

$$\Delta = \begin{vmatrix} 3 + j3 & -j3 & 3 \\ -j3 & 14 + j3 & 6 \\ 3 & 6 & 9 - j4 \end{vmatrix} = 348 + j48 = 351 \angle 7.86^\circ$$

$$N_1 = \begin{vmatrix} 10 & -j3 & 3 \\ 0 & 14 + j3 & 6 \\ j5 & 6 & 9 - j4 \end{vmatrix} = 1155 - j500 = 1260 \angle -23.40^\circ$$

$$N_2 = \begin{vmatrix} 3 + j3 & 10 & 3 \\ -j3 & 0 & 6 \\ 3 & j5 & 9 - j4 \end{vmatrix} = 435 + j180 = 470 \angle 22.50^\circ$$

$$N_3 = \begin{vmatrix} 3 + j3 & -j3 & 10 \\ -j3 & 14 + j3 & 0 \\ 3 & 6 & j5 \end{vmatrix} = -675 - j60 = 678 \angle -174.90^\circ$$

The mesh currents then become:

$$I_1 = \frac{1260 \angle -23.40^\circ}{351 \angle 7.86^\circ} = 3.59 \angle -31.26^\circ = 3.07 - j1.863$$

$$I_2 = \frac{470 \angle 22.50^\circ}{351 \angle 7.86^\circ} = 1.34 \angle 14.64^\circ = 1.298 + j0.339$$

$$I_3 = \frac{678 \angle -174.90^\circ}{351 \angle 7.86^\circ} = 1.93 \angle -182.76^\circ = -1.93 + j0.093$$

APPENDIX III

HYPERBOLIC FUNCTIONS

Difficulty in the use of hyperbolic functions often arises because the student fails to note the close similarity between them and the more familiar circular functions. When mention is made of a function such as $\cos x$ any engineer knows instinctively what is meant; but when the student meets, for the first time, the function $\cosh x$ or $\sinh x$ he is immediately afflicted with hysteria. An attempt will be made here to place these two types of functions upon the same footing. In the first place a source of confusion arises because the circular functions are most often referred to degrees. Immediately one wishes to know what is analogous to degrees in hyperbolic functions. There is no analogous unit. By $\cos x$, $\cosh x$, etc., are meant certain functions of a number, x, which is given as so many *radians*.

A beginning is made by writing down four functions out of which subsequent material will be developed.

Circular Functions		*Hyperbolic Functions*	
(1)	$\sin x$	(3)	$\sinh x$
(2)	$\cos x$	(4)	$\cosh x$

These items are merely four functions of the quantity x. Their definitions are:

$$\sin x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{2j} \qquad \sinh x = \frac{\epsilon^{x} - \epsilon^{-x}}{2}$$

$$\cos x = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2} \qquad \cosh x = \frac{\epsilon^{x} + \epsilon^{-x}}{2}$$

From these equations it is easily seen that, from all appearances, the hyperbolic functions are simpler than the circular functions.

It is known from past experience that $\sin x$ and $\cos x$ are periodic as x is increased from zero to plus or minus infinity. Presumably, for the present, it is not known how $\sinh x$ and $\cosh x$ behave, but their characteristics are easily deduced from the above equations. Obviously both $\sinh x$ and $\cosh x$ increase from certain low values to infinity as x is increased. When x is zero $\sinh x$ has the value zero and $\cosh x$ has the

value unity. Thus it is found that while circular functions are periodic, hyperbolic functions are not. Tables can be made for both types of functions and any one of the four quantities mentioned can be found by looking for the appropriate value of x. As an illustration, assume that the values of sin 5 and sinh 5 are to be found. Since circular-function tables are usually compiled in terms of degrees instead of radians it is necessary to change the 5 radians to degrees, which is done by multiplying by 57.3. Thus the procedure is to look up $\sin 286.5°$ $[= \sin(-73.5°) = -\sin 73.5° = -0.959]$. Sinh 5 can be found directly from the tables and is 74.2.

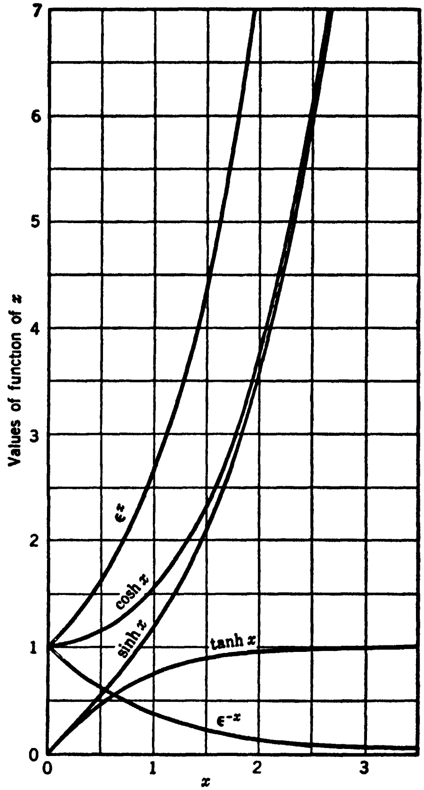

FIG. A-6. Variation of hyperbolic and exponential functions.

With this brief introduction, a development concerning hyperbolic functions alone will be undertaken. In Fig. A-6 are shown plots of ϵ^x and ϵ^{-x} for positive values of x. Their sum and difference divided by 2 are also shown, thus giving a picture of the two fundamental functions. Other functions are defined on the basis of these two in the same manner as in circular functions. Thus there exist tanh x, coth x, etc.

$$\tanh x = \frac{1}{\coth x} = \frac{\sinh x}{\cosh x} = \frac{\epsilon^x - \epsilon^{-x}}{\epsilon^x + \epsilon^{-x}}$$

This function begins at zero for $x = 0$ and increases asymptotically to unity as x increases. For the present purposes these four functions will be the only ones necessary.

There exists a whole array of identities involving these functions in much the same way as those involving circular functions. For instance, it will be desirable to know other means of expressing the fol-

lowing combinations:

$$\sinh a \cosh b$$
$$\cosh a \sinh b$$
$$\sinh a \sinh b$$
$$\cosh a \cosh b$$

Using fundamental definitions, the first product becomes

$$\sinh a \cosh b = \frac{\epsilon^a - \epsilon^{-a}}{2} \cdot \frac{\epsilon^b + \epsilon^{-b}}{2}$$

$$= \frac{\epsilon^{a+b} + \epsilon^{a-b} - \epsilon^{-a+b} - \epsilon^{-a-b}}{4}$$

$$= \frac{\epsilon^{a+b} - \epsilon^{-(a+b)}}{4} + \frac{\epsilon^{a-b} - \epsilon^{-(a-b)}}{4}$$

$$= \frac{\sinh (a + b) + \sinh (a - b)}{2} \qquad \text{[A-10]}$$

Interchanging a and b in equation A-10 will give the second product. A close analogy to equation A-10 can be obtained for this product if use is made of the fact that $\sinh x = - \sinh (-x)$. This relation can be shown as follows:

$$\sinh x = \frac{\epsilon^x - \epsilon^{-x}}{2}$$

$$\sinh (-x) = \frac{\epsilon^{-x} - \epsilon^x}{2}$$

$$= - \frac{\epsilon^x - \epsilon^{-x}}{2} = - \sinh x$$

A similar procedure will show that the $\cosh x$ is not changed in value by changing the sign of x. Thus

$$\cosh a \sinh b = \frac{\sinh (a + b) - \sinh (a - b)}{2} \qquad \text{[A-11]}$$

Similarly

$$\sinh a \sinh b = \frac{\cosh (a + b) - \cosh (a - b)}{2} \qquad \text{[A-12]}$$

and

$$\cosh a \cosh b = \frac{\cosh (a + b) + \cosh (a - b)}{2} \qquad \text{[A-13]}$$

Thus four identities have already been found which are similar in form to the corresponding ones in circular functions. From these can be obtained

$$\cosh (a + b) \,[= (\text{A–12}) + (\text{A–13})] = \cosh a \cosh b + \sinh a \sinh b \tag{A–14}$$

$$\cosh (a - b) \,[= (\text{A–13}) - (\text{A–12})] = \cosh a \cosh b - \sinh a \sinh b \tag{A–15}$$

$$\sinh (a + b) \,[= (\text{A–10}) + (\text{A–11})] = \sinh a \cosh b + \cosh a \sinh b \tag{A–16}$$

$$\sinh (a - b) \,[= (\text{A–10}) - (\text{A–11})] = \sinh a \cosh b - \cosh a \sinh b \tag{A–17}$$

By making use of the fundamental definitions the following useful relations can also be easily proved:

$$\cosh^2 x - \sinh^2 x = 1 \tag{A–18}$$

$$\cosh^2 x + \sinh^2 x = \cosh 2x \tag{A–19}$$

In order to determine certain expanded forms of equations for hyperbolic functions of complex quantities, two relations between circular and hyperbolic functions are needed. From the basic definitions

$$\sinh jx = \frac{j(\epsilon^{jx} - \epsilon^{-jx})}{2j} = j \sin x$$

Also

$$\cosh jx = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2} = \cos x$$

Using these, equations A–14, A–15, A–16, and A–17 are readily transformed to

$$\begin{aligned} \cosh (a + jb) &= \cosh a \cosh jb + \sinh a \sinh jb \\ &= \cosh a \cos b + j \sinh a \sin b \end{aligned} \tag{A–20}$$

$$\cosh (a - jb) = \cosh a \cos b - j \sinh a \sin b \tag{A–21}$$

$$\begin{aligned} \sinh (a + jb) &= \sinh a \cosh jb + \cosh a \sinh jb \\ &= \sinh a \cos b + j \cosh a \sin b \end{aligned} \tag{A–22}$$

$$\sinh (a - jb) = \sinh a \cos b - j \cosh a \sin b \tag{A–23}$$

It is thus seen that hyperbolic functions of complex quantities can be expressed as complex quantities, $M + jN$, and if a and b are known any

one of the expressions (equations A–20, A–21, A–22, and A–23) can be calculated.

Useful forms of equations A–20 and A–22 result when expressed in polar form. From equation A–22, using equation A–18

$$\sinh(a + jb) = \sinh a \cos b + j \cosh a \sin b$$
$$= \sqrt{\sinh^2 a \cos^2 b + \cosh^2 a \sin^2 b} \;\Big/ \tan^{-1} \frac{\sin b \cosh a}{\cos b \sinh a}$$

$$= \sqrt{\sinh^2 a(1 - \sin^2 b) + \sin^2 b(1 + \sinh^2 a)} \;\Big/ \tan^{-1} \frac{\sin b \cosh a}{\cos b \sinh a}$$

$$= \sqrt{\sinh^2 a + \sin^2 b} \;\Big/ \tan^{-1} \frac{\sin b \cosh a}{\cos b \sinh a} \qquad \text{[A–24]}$$

Similarly

$$\cosh(a + jb) = \sqrt{\cosh^2 a \cos^2 b + \sinh^2 a \sin^2 b} \;\Big/ \tan^{-1} \frac{\sin b \sinh a}{\cos b \cosh a}$$

$$= \sqrt{(1 + \sinh^2 a)\cos^2 b + \sinh^2 a(1 - \cos^2 b)} \;\Big/ \tan^{-1} \frac{\sin b \sinh a}{\cos b \cosh a}$$

$$= \sqrt{\sinh^2 a + \cos^2 b} \;\Big/ \tan^{-1} \frac{\sin b \sinh a}{\cos b \cosh a} \qquad \text{[A–25]}$$

It is a different matter, however, if, instead of a and b being given, the function of $(a + jb)$ is given and it is required to find a and b. For example, assume that $\tanh(a + jb) = M + jN$ is given in which M and N are known. Consider the following development:

$$M + jN = \tanh(a + jb) = \frac{\sinh(a + jb)}{\cosh(a + jb)}$$
$$= \frac{\sinh a \cos b + j \cosh a \sin b}{\cosh a \cos b + j \sinh a \sin b}$$

Rationalizing

$$M + jN =$$
$$\frac{\sinh a \cosh a \cos^2 b + \sinh a \cosh a \sin^2 b - j \sinh^2 a \cos b \sin b + j \cosh^2 a \cos b \sin b}{\cosh^2 a \cos^2 b + \sinh^2 a \sin^2 b}$$

Using equation A–18 and the relation presented in the development of equation A–25

$$M + jN \qquad \frac{\sinh a \cosh a + j \sin b \cos b}{\sinh^2 a + \cos^2 b}$$

Equating real and quadrature parts and making use of equation A–16

$$M = \frac{\sinh a \cosh a}{\sinh^2 a + \cos^2 b} = \frac{\sinh 2a}{2(\sinh^2 a + \cos^2 b)} \qquad \text{[A–26]}$$

$$N = \frac{\sin b \cos b}{\sinh^2 a + \cos^2 b} = \frac{\sin 2b}{2(\sinh^2 a + \cos^2 b)} \qquad \text{[A–27]}$$

From equations A–24 and A–25

$$M^2 + N^2 = |M + jN|^2 = |\tanh(a + jb)|^2 = \left|\frac{\sinh^2(a + jb)}{\cosh^2(a + jb)}\right|$$

$$= \frac{\sinh^2 a + \sin^2 b}{\sinh^2 a + \cos^2 b}$$

and

$$1 + M^2 + N^2 = \frac{\sinh^2 a + \cos^2 b + \sinh^2 a + \sin^2 b}{\sinh^2 a + \cos^2 b}$$

$$= \frac{2\sinh^2 a + 1}{\sinh^2 a + \cos^2 b} = \frac{\cosh 2a}{\sinh^2 a + \cos^2 b} \qquad \text{[A–28]}$$

from equations A–18 and A–19. Similarly

$$1 - (M^2 + N^2) = \frac{\sinh^2 a + \cos^2 b - \sinh^2 a - \sin^2 b}{\sinh^2 a + \cos^2 b}$$

$$= \frac{\cos 2b}{\sinh^2 a + \cos^2 b} \qquad \text{[A–29]}$$

Using equations A–26 and A–28 gives

$$\tanh 2a = \frac{2M}{1 + M^2 + N^2} \qquad \text{[A–30]}$$

and using equations A–27 and A–29

$$\tan 2b = \frac{2N}{1 - (M^2 + N^2)} \qquad \text{[A–31]}$$

In using these equations it must be noted that the function in equation A–31 is multivalued, and in order to fix b definitely some additional information must be available. In particular, if the signs of the numerator and denominator of equation A–31 are known separately, then the quadrant of $2b$ is definitely fixed. The sign of the numerator is found, of course, from the original expression, $\tanh(a + jb) = M + jN$. The sign of the denominator takes care of itself because of the fact that both

M and N appear in it squared. As an illustration, refer to Art. 60 where

$$\tan 2b = \frac{-2 \times 1.54}{1 - 2.479}$$

or the ratio of two negative quantities. Accordingly, since only angles in the third quadrant have tangents with a negative numerator and negative denominator, the angle $2b$ lies in the third quadrant.

Where $\cosh(a + jb)$ is given, with the requirement that a and b be found, it will be necessary to change to the hyperbolic tangent. This can be effected as follows:

$$\sinh^2 (a + jb) = \cosh^2 (a + jb) - 1$$

$$\sinh (a + jb) = \sqrt{\cosh^2 (a + jb) - 1}$$

and

$$\tanh (a + jb) = \frac{\sqrt{\cosh^2 (a + jb) - 1}}{\cosh (a + jb)} \qquad [A-32]$$

When $\sinh(a + jb)$ is given, there is obtained

$$\tanh (a + jb) = \frac{\sinh (a + jb)}{\sqrt{\sinh^2 (a + jb) + 1}} \qquad [A-33]$$

Equations A–32 and A–33, although relatively simple in appearance, are somewhat difficult to use. The student is advised to work out an example in order to become familiar with the procedure.

APPENDIX IV

ALTERNATIVE SOLUTION FOR Z_0 AND γ[1]

Expressions for Z_0 and γ in terms of distributed line parameters were derived in Chapter V. An alternative method of obtaining these same relations can be found from equations 4–1 and 4–7, which give Z_0 and γ for finite sections of a line, by allowing the length of the section to become infinitesimally small. The equations for Z_0 and γ for finite sections of a line are

$$Z_0 = \sqrt{Z_1 Z_2 + \frac{Z_1^2}{4}} \qquad [4\text{–}1]$$

$$\epsilon^{\gamma} = 1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2} + \frac{Z_1^2}{4Z_2^2}} \qquad [4\text{–}7]$$

It has been stated previously that since these equations apply to sections of finite length they cannot be exactly correct. It is to be expected then that, if the length of the section is allowed to become vanishingly small, Z_0 and γ should approach their true values as given by a line with distributed parameters.

Let it be recalled that Z_1 is the series impedance and Z_2 is the shunt impedance of the finite section of line, whereas z is the series impedance and y is the shunt admittance per mile. Since the length of line is to be allowed to become very small, its value will be given by Δx. From the definition of Δx and z it is clear that $Z_1 = z\Delta x$, and $1/Z_2$, the admittance of the section, will be $y\Delta x$. Let these values be substituted into equation 4–1.

$$Z_0 = \sqrt{\frac{z\Delta x}{y\Delta x} + \frac{z^2(\Delta x)^2}{4}} \qquad [A\text{–}34]$$

As Δx approaches zero, Z_0 becomes $\sqrt{z/y}$, which is equation 5–16 as developed in Chapter V.

For the evaluation of γ, equation 4–7 will first be expanded by the

[1] Everitt, W. L., *Communication Engineering*, New York, McGraw-Hill Book Co., 1937.

binomial theorem. Thus the radical becomes

$$\sqrt{\frac{Z_1}{Z_2}+\left(\frac{Z_1}{2Z_2}\right)^2}=\left(\frac{Z_1}{Z_2}\right)^{1/2}+\frac{1}{2}\left(\frac{Z_1}{Z_2}\right)^{-1/2}\cdot\left(\frac{Z_1}{2Z_2}\right)^2+\cdots \qquad \text{[A-35]}$$

and equation 4–7 can be written

$$\epsilon^\gamma = 1+\left(\frac{Z_1}{Z_2}\right)^{1/2}+\frac{1}{2}\left(\frac{Z_1}{Z_2}\right)+\frac{1}{8}\left(\frac{Z_1}{Z_2}\right)^{3/2}+\cdots \qquad \text{[A-36]}$$

Since $Z_1/Z_2 = zy(\Delta x)^2$

$$\epsilon^\gamma = 1+\sqrt{zy}\,(\Delta x)+\tfrac{1}{2}(zy)(\Delta x)^2+\tfrac{1}{8}(zy)^{3/2}(\Delta x)^3+\cdots \qquad \text{[A-37]}$$

Compare this expression for ϵ^γ with the usual exponential series.

$$\epsilon^\gamma = 1+\gamma+\tfrac{1}{2}\gamma^2+\tfrac{1}{6}\gamma^3+\cdots \qquad \text{[A-38]}$$

Since Δx is being allowed to approach zero, the terms in $(\Delta x)^3$ and higher may be neglected, and the others equated, term by term. Thus the series for the first three terms would be identical if γ were made equal to $\sqrt{zy}(\Delta x)$, and this γ would be the propagation constant for the infinitesimally short section. Assume that there is a very large number, n, of these short sections per mile. Then the attenuation constant per mile is $\gamma_m = n\gamma = n\sqrt{zy}(\Delta x)$. However, if n is the number of sections of length Δx per mile, then $n\Delta x = 1$, and $\gamma_m = \sqrt{zy}$, or in the more usual notation without the subscript

$$\gamma = \sqrt{zy}$$

APPENDIX V

ALTERNATIVE DERIVATION OF EQUATIONS 6-5 AND 6-6[1]

The derivation of certain general equations of a transmission line by a method which provides a means of partly visualizing the phenomenon which takes place is presented here. Reflection, as it occurs on a transmission line, actually consists of a sequence of events: (1) A wave front proceeds along the line and when it arrives at the termination, provided the termination is other than Z_0, part of its energy is absorbed and part is reflected; (2) the reflected wave travels back along the line until it meets the sending-end impedance, where part of its energy is further absorbed and the remainder is again reflected toward the receiving end. This process of reflection continues until the loss in the line and in the terminating impedances reduces the energy of the wave to zero. The resultant line current or voltage, as measured by instruments, is the sum of all of these reflected waves. Obviously, since these waves are continually subject to the effects of attenuation and phase constants, the phenomenon is impossible to visualize clearly.

A simplification which will provide an opportunity of forming a physical picture of reflection, even though it is incorrect, is desirable. It is known that if a line is terminated in its characteristic impedance Z_0 it acts as an infinite line, propagation takes place in one direction only, and its current and voltage at any point may be readily calculated. Hence, it is proposed to fashion two correctly terminated lines out of the given line together with its improper termination. These two lines can then be treated separately, their currents and voltages at any point found, and by the superposition theorem these currents or voltages can be added together to obtain the true current or voltage.

Let it be assumed that there are provided (1) a generator of emf E_g and impedance Z_g, (2) a line of characteristic impedance Z_0 and propagation γl, and (3) a receiving-end impedance Z_r. In Fig. A-7 are shown the given line a and the successive stages involved in the process of breaking it up into two correctly terminated lines.

In Fig. A-7b the generator impedance Z_g has been replaced by two impedances (Z_g-Z_0) and Z_0. Likewise the receiving-end impedance Z_r

[1] Everitt, W. L., *Communication Engineering*, p. 132, New York, McGraw-Hill Book Co., 1937.

has been replaced by the impedances $(Z_r\text{-}Z_0)$ and Z_0. Inspection of the diagram will show this break-up to be perfectly legitimate. Since the compensation theorem, Art. 23, allows an impedance drop to be replaced by an equivalent generated emf, the circuit of Fig. A-7*c* is obtained by substituting the emf's E_a and E_b for the impedance drops

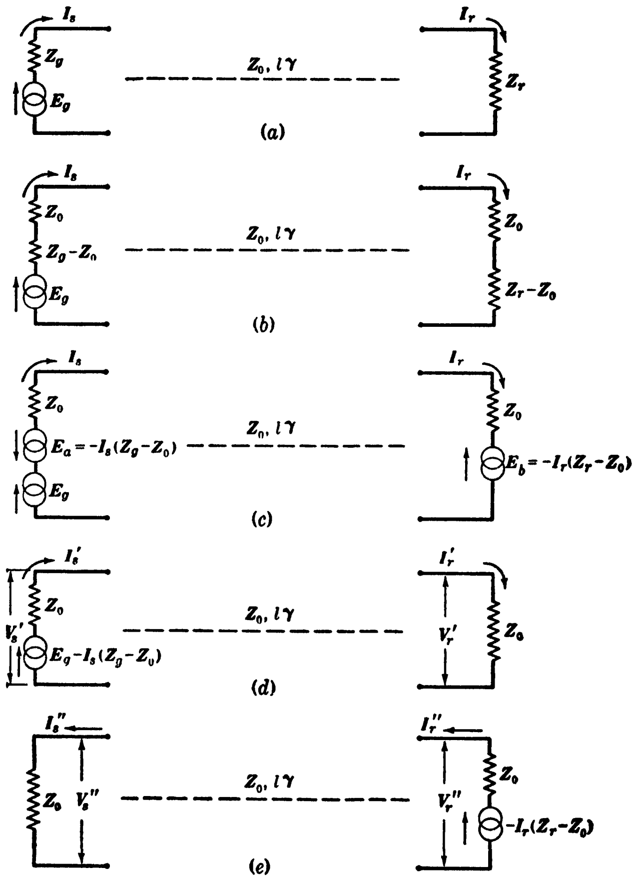

FIG. A-7. Transformation of a given line into two correctly terminated lines.

$I_s(Z_g\text{-}Z_0)$ and $I_r(Z_r\text{-}Z_0)$. The two circuits of Figs. A-7*d* and A-7*e* are obtained from Fig. A-7*c* and each of them will yield part of the final result. It will be noted that in *d* a correctly terminated line is transmitting toward the right and its generator is made up partly of the original generator and partly of the fictitious generator added because of the drop through the impedance $(Z_g\text{-}Z_0)$. The diagram in *e* represents a

correctly terminated line transmitting toward the left, and its generator is a fictitious one owing to the impedance drop $I_r(Z_r\text{-}Z_0)$. In neither of these lines will there be any reflection. After the current and voltage at any point of the line are calculated due to each circuit separately the results can be superposed to give the actual existing values. It must be realized, however, that obviously the picture of only two waves, one to the right and one to the left, is wrong. However, the great advantage of the method lies in the fact that it often enables problems to be handled by a graphical or step-by-step analytical method.

The following notation will be used in the analytical derivation of equations 6–5 and 6–6:

V_s'	is	the	line	voltage	at	sending	end	due	to	sending-end	generator
V_r'	"	"	"	"	"	receiving	"	"	"	" "	"
V_s''	"	"	"	"	"	sending	"	"	"	receiving-end	"
V_r''	"	"	"	"	"	receiving	"	"	"	" "	"
I_s'	"	"	"	current	"	sending	"	"	"	sending-end	"
I_r'	"	"	"	"	"	receiving	"	"	"	" "	"
I_s''	"	"	"	"	"	sending	"	"	"	receiving-end	"
I_r''	"	"	"	"	"	receiving	"	"	"	" "	"

The following equations can be written, based upon Figs. A–7*d* and A–7*e*. On a correctly terminated line $V_r = V_s\epsilon^{-\gamma l}$ and $I_r = I_s\epsilon^{-\gamma l}$, from equation 4–8. The propagation constant for the entire line is γl.

$$I_s = I_s' + I_s'' \qquad [A\text{–}39]$$

$$I_r = I_r' + I_r'' \qquad [A\text{–}40]$$

From Fig. A–7*d*,

$$I_s' = \frac{E_g - I_s(Z_g - Z_0)}{2Z_0} \qquad [A\text{–}41]$$

$$I_r' = I_s'\epsilon^{-\gamma l} = \frac{[E_g - I_s(Z_g - Z_0)]\epsilon^{-\gamma l}}{2Z_0} \qquad [A\text{–}42]$$

From Fig. A–7*e*,

$$I_r'' = \frac{-I_r(Z_r - Z_0)}{2Z_0} \qquad [A\text{–}43]$$

$$I_s'' = \frac{-I_r(Z_r - Z_0)\epsilon^{-\gamma l}}{2Z_0} \qquad [A\text{–}44]$$

Let equations A–41 to A–44 be substituted into equations A–39 and A–40.

$$I_s = \frac{E_g - I_s(Z_g - Z_0)}{2Z_0} - \frac{I_r(Z_r - Z_0)\epsilon^{-\gamma l}}{2Z_0} \qquad [A\text{–}45]$$

and

$$I_r = \frac{[E_g - I_s(Z_g - Z_0)]\epsilon^{-\gamma l}}{2Z_0} - \frac{I_r(Z_r - Z_0)}{2Z_0} \qquad \text{[A-46]}$$

or

$$I_s = \frac{E_g - I_s(Z_g - Z_0) - I_r(Z_r - Z_0)\epsilon^{-\gamma l}}{2Z_0} \qquad \text{[A-47]}$$

and

$$I_r = \frac{E_g\epsilon^{-\gamma l} - I_s(Z_g - Z_0)\epsilon^{-\gamma l} - I_r(Z_r - Z_0)}{2Z_0} \qquad \text{[A-48]}$$

Equations A-47 and A-48 may be treated as simultaneous equations and solved for I_r and I_s. The solution yields

$$I_r = \frac{2E_gZ_0}{(Z_0 + Z_r)(Z_g + Z_0)\epsilon^{\gamma l} + (Z_0 - Z_r)(Z_g - Z_0)\epsilon^{-\gamma l}} \qquad \text{[A-49]}$$

$$I_s = \frac{E_g[(Z_0 + Z_r)\epsilon^{\gamma l} + (Z_0 - Z_r)\epsilon^{-\gamma l}]}{(Z_0 + Z_r)(Z_g + Z_0)\epsilon^{\gamma l} + (Z_0 - Z_r)(Z_g - Z_0)\epsilon^{-\gamma l}} \qquad \text{[A-50]}$$

It is desirable to have these equations expressed in terms of V_s, the voltage across the line, instead of the generator emf E_g; so by making $E_g = V_s$ and $Z_g = 0$ the input voltage across the line V_s will appear in the equations. Thus

$$I_r = \frac{2V_sZ_0}{(Z_0 + Z_r)Z_0\epsilon^{\gamma l} + (Z_0 - Z_r)(-Z_0)\epsilon^{-\gamma l}}$$

$$= \frac{V_sZ_0}{Z_0^2\dfrac{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}{2} + Z_0Z_r\dfrac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{2}}$$

$$= \frac{V_s}{Z_0 \sinh \gamma l + Z_r \cosh \gamma l} \qquad \text{[A-51]}$$

This equation is the same as equation 6-5, where $\sqrt{zy}S$ takes the place of γl. Likewise

$$I_s = \frac{V_s[(Z_0 + Z_r)\epsilon^{\gamma l} + (Z_0 - Z_r)\epsilon^{-\gamma l}]}{(Z_0 + Z_r)Z_0\epsilon^{\gamma l} + (Z_0 - Z_r)(-Z_0)\epsilon^{-\gamma l}}$$

which, on rearranging and making substitutions of the hyperbolic functions, becomes

$$I_s = \frac{V_s(Z_0 \cosh \gamma l + Z_r \sinh \gamma l)}{Z_0(Z_r \cosh \gamma l + Z_0 \sinh \gamma l)} \qquad \text{[A-52]}$$

This corresponds to equation 6-6.

The preceding analysis easily leads to a reasonably satisfactory physical picture of the conditions on an open- (or a short-) circuited transmission line if α may be considered as zero. For this case, since there is no decrease in the amplitude of the wave as it progresses along the line, the superposition of the two waves traveling in opposite directions is comparatively easy to visualize. The wave traveling toward the receiver end of the line will be considered as the direct wave and the one sent back from the receiver end as the reflected wave. The voltage (or current) along the line at any point may be considered as the vector sum of these two waves.

As an illustration, let the voltage distribution on a short-circuited line be considered. It should be remembered that, with respect to time, vectors rotate counterclockwise and with respect to distance traversed they rotate clockwise. For this illustration, since it is the distribution

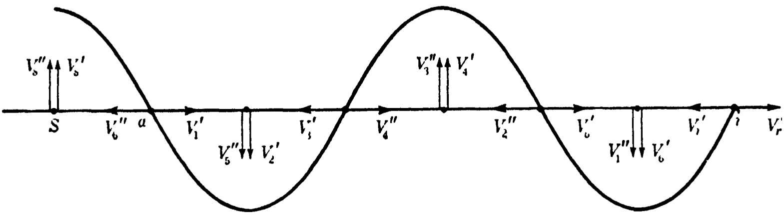

FIG. A-8. Representation of reflection on a short-circuited line.

along the line at a particular instant which is of interest, the vectors concerned will rotate clockwise as long as one is progressing in the direction of transmission. The voltage vector at the receiving end of the line undergoes a phase shift of 180° because only in this way can the sum of the two vectors produce the necessary boundary condition at a short circuit, that of zero voltage. The vector, after reversal, may be considered at successive points along the line back to the sending end, its rotation always being clockwise, and at certain points the vectors representing the direct and reflected waves are found to be 180° out of phase just as they are at the receiving end. Thus there are positions along the line of zero voltage, and also positions of maximum voltage where the vectors are added linearly.

The general situation can be seen more clearly in Fig. A-8, where a 1¾-wavelength line is considered. V_s' represents the input voltage at the sending end of the line. As one progresses along the line the direct wave vector takes positions at every quarter-wavelength, as shown by the vectors V_1', V_2', $\cdots$ V_r'. At the receiver end a complete reversal of V_r' takes place in order to fulfill the condition of zero voltage at that

point. This new vector is designated as V_r'', and its successive positions at points separated by quarter wavelengths along the line are represented by V_1'', V_2'', $\cdots$ V_s''. As stated previously, the voltage at any point of the line will be given by the vector sum of the two vectors at that point. Thus at S the voltage is a maximum, $V_s' + V_s''$, at a it is zero, and so on along the line until zero voltage is again attained at r. If the negative loops were reversed to indicate merely meter readings then the similarity between this curve and the curve of V for the short-circuited line of Fig. 6-13 could easily be seen.

APPENDIX VI

MAXWELL'S EQUATIONS

Maxwell's equations are usually written in vector form. However, for the purposes of certain derivations presented in this text, the equations which are of immediate use will be written in *scalar form* because the use of vector notation here serves no useful purpose. Written in this manner, the two equations will appear as six. The first group of three refers to the relation between electric currents and the resultant magnetic fields, and the second group refers to the relation between the electric field and the time rate of change of current, or magnetic field. It will be seen presently that these equations are merely general statements of the circuital law of magnetism and Faraday's emf law respectively, expressed in differential forms. The equations are especially useful in the treatment of the propagation of electromagnetic waves, and they find application not only in radio but also in many problems of wire or guided transmission. Many books treat of their derivation and reference should be made to them for greater completeness.[1]

Derivation of Maxwell's Equations in Cartesian Coordinates. (*a*) *Maxwell's First Law.* The first set of equations is based on the circuital law of magnetism which in equation form and in rationalized units is

$$\oint H \cdot dl = I \qquad \text{[A–53]}$$

[1] Doherty, R. E., and E. G. Keller, *Mathematics of Modern Engineering*, New York, John Wiley & Sons, 1947. Bronwell, A. B., and R. E. Beam, *Theory and Application of Microwaves*, New York, McGraw-Hill Book Co., 1947. Hund, A., *Phenomena in High Frequency Systems*, New York, McGraw-Hill Book Co., 1936. King, R. W. P., *Electromagnetic Engineering*, Vol. 1, New York, McGraw-Hill Book Co., 1945. Marchand, N., *Ultrahigh Frequency Transmission and Radiation*, New York, John Wiley & Sons, 1947. McIlwain, K., and J. G. Brainerd, *High Frequency Alternating Currents*, New York, John Wiley & Sons, 1939. Page, L., *Introduction to Theoretical Physics*, New York, Van Nostrand Co., 1930. Ramo, S., and J. R. Whinnery, *Fields and Waves in Modern Radio*, New York, John Wiley & Sons, 1944. Schelkunoff, S. A., *Electromagnetic Waves*, New York, Van Nostrand Co., 1943. Skilling, H. H., *Fundamentals of Electric Waves*, New York, John Wiley & Sons, 1948. Stratton, J. A., *Electromagnetic Theory*, New York, McGraw-Hill Book Co., 1941. Ware, L.A., *Elements of Electromagnetic Waves*, New York, Pitman Publ. Corp., 1949. Sarbacher, R. I., and W. A. Edson, *Hyper and Ultrahigh Frequency Engineering*, New York, John Wiley & Sons, 1943.

where H is the magnetic-field intensity in amperes per meter and l is the displacement or distance along the closed path which encircles the current. In this derivation, the current I is expressed in amperes and the displacement l is expressed in meters. (It is understood that in general H is a function of both time and space.)

If all space is assumed to be filled with electric currents and the associated magnetic fields, it is a relatively simple matter to establish

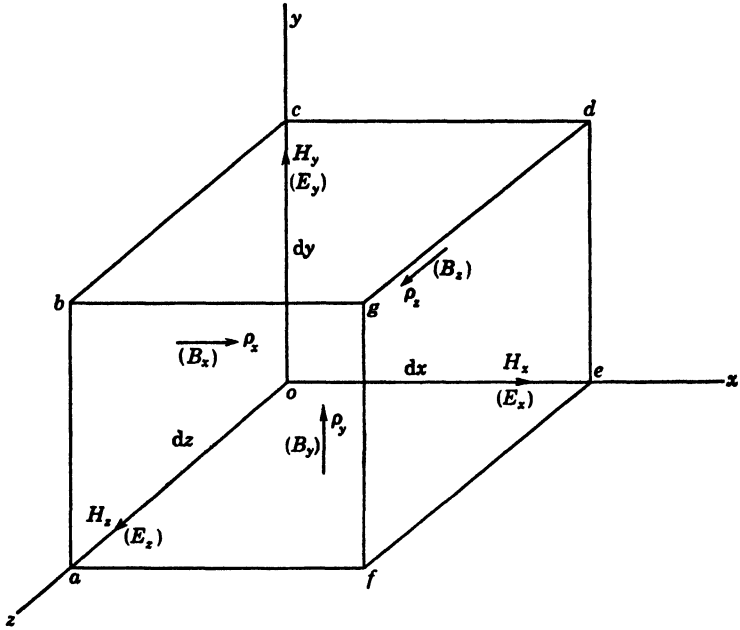

FIG. A–9. Element of volume in the electromagnetic field. Cartesian coordinates.

the relationships between the space variations in H and the current densities which exist at any point in space. This set of relationships is sometimes referred to as Maxwell's first law. Let Fig. A–9 represent an elemental section of space filled with electric and magnetic fields, and with the associated currents. Also let ρ_x, ρ_y, and ρ_z represent the current densities in the x, y, and z directions respectively. The magnetic-field intensities along the x, y, and z axes respectively will be represented by H_x, H_y, and H_z. The general principle involved in the establishment of the first equation to be considered can be seen by treating only one surface of the element of volume. Assume that the area *ocbao* is selected. Through this area the total current is

$$I_x = \rho_x dydz = \rho_x dA \qquad \text{[A–54]}$$

Around the boundary of this surface there exist magnetic intensity or H vectors, two of which are indicated in Fig. A–9, namely, H_y along the dy path and H_z along the dz path.

The magnetic potential drops around the *ocbao* loop taken individually are

$$dU_{1(\text{along } oc)} = H_y dy$$

$$dU_{2(\text{along } cb)} = H_z dz + \frac{\partial(H_z dz)}{\partial y} dy = H_z dz + \frac{\partial H_z}{\partial y} dydz$$

$$dU_{3(\text{along } ba)} = -\left[H_y dy + \frac{\partial(H_y dy)}{\partial z} dz\right] = -H_y dy - \frac{\partial H_y}{\partial z} dydz$$

$$dU_{4(\text{along } ao)} = -H_z dz$$

In arriving at these expressions it is of course recognized that dz is not a function of y and neither is dy a function of z. The four magnetic potential drops are to be taken in the *ocbao* direction around the loop since $+I_x$ establishes H vectors in this direction around the loop in accordance with the right-hand rule.

From the circuital law of magnetism (equation A–53), it is plain that

$$\oint H \cdot dl = dU_1 + dU_2 + dU_3 + dU_4$$

$$= \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) dydz = \rho_x dydz \qquad \text{[A–55]}$$

or

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \rho_x \qquad \text{[A–56]}$$

In this equation ρ_x the current density existing over the $dydz$ face and directed along the x axis is made up of two parts. One is the conduction current density gE_x, where g is the specific conductance per unit volume and E_x is the x component of the electric-field intensity in volts per meter. The other is the displacement current density, $\partial D_x/\partial t$, where D_x is the electric flux density. Since $D = \varepsilon E$, where ε is the permittivity of the medium, the total current density may be written

$$\rho_x = gE_x + \varepsilon \frac{\partial E_x}{\partial t}$$

Equation A–56 now becomes

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = gE_x + \varepsilon \frac{\partial E_x}{\partial t} \qquad \text{[A–57]}$$

In an exactly similar manner two other equations, for the remaining

two coordinate directions, may be derived. They are

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = gE_y + \varepsilon \frac{\partial E_y}{\partial t} \qquad \text{[A-58]}$$

and

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = gE_z + \varepsilon \frac{\partial E_z}{\partial t} \qquad \text{[A-59]}$$

These three equations together make up the expression of one of Maxwell's laws. They express three of the necessary relations which must always exist between H and E in the electromagnetic field.

(*b*) *Maxwell's Second Law.* Equations A–57, A–58, and A–59 are based on the circuital law of magnetism. Another set of three equations based on Faraday's law will now be derived. Again Fig. A–9 will be used with the ρ's replaced by flux densities B and with the H's giving place to corresponding electric intensities (E's) expressed in volts per meter.

Consider the area *ocbao* and assume that the flux density B_x is decreasing so that its derivative with respect to time is negative. Also assume for the moment that the boundaries of the area are fine wires, with practically infinite resistance if we wish, in which emf's are induced by the time rate of change of B_x. The decrease of flux through the area will induce a voltage e in the wire boundary which will be in the sense *ocbao*. The magnitude of this voltage is given by Faraday's law to be

$$e = \oint E \cdot dl = -\frac{d\phi_x}{dt} = -\frac{\partial B_x}{\partial t}\, dydz \qquad \text{[A-60]}$$

where E is the electric intensity vector
- l is the displacement directed along the periphery of loop *oabco*
- ϕ_x is the magnetic flux crossing the *dydz* surface
- B_x is the flux density at the *dydz* surface

The minus sign accounts for the fact that the voltage induced in the *ocbao* loop is measured by the time rate of decrease of magnetic flux through the loop.

The electric potential differences around the closed path *ocbao* (taken in the right-hand screw direction around $+B_x$) are individually

$$dv_{1(\text{along } oc)} = E_y dy$$

$$dv_{2(\text{along } cb)} = E_z dz + \frac{\partial(E_z dz)}{\partial y} dy = E_z dz + \frac{\partial E_z}{\partial y} dydz$$

$$dv_{3(\text{along } ba)} = -\left[E_y dy + \frac{\partial(E_y dy)}{\partial z} dz\right] = -E_y dy - \frac{\partial E_y}{\partial z} dydz$$

$$dv_{4(\text{along } ao)} = -E_z dz$$

?rom equation A-60 it is seen that

$$e = dv_1 + dv_2 + dv_3 + dv_4 = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) dydz = -\frac{\partial B_x}{\partial t} dydz \quad \text{[A-61]}$$

Recognizing that $B_x = \mu_0 H_x$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_0 \frac{\partial H_x}{\partial t} \quad \text{[A-62]}$$

If the same procedure is applied to faces *dxdz* and to *dxdy* respectively, we find that

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu_0 \frac{\partial H_y}{\partial t} \quad \text{[A-63]}$$

and

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_0 \frac{\partial H_z}{\partial t} \quad \text{[A-64]}$$

Equations A-62, A-63, and A-64 express three more necessary relations which always exist between the H and E vectors in the electromagnetic field. Taken collectively, they are sometimes referred to as Maxwell's second law.

Equations A-57, A-58, A-59, A-62, A-63, and A-64 have been derived in rationalized mks units and hence are most directly applicable in this system of units where

$\mu_0 = 4\pi \times 10^{-7}$ henry/meter, the permeability of free space
$\varepsilon = \epsilon_0\epsilon_r$, the permittivity expressed in farads/meter
$\epsilon_0 = 1/(36\pi \times 10^9)$ farad/meter, the permittivity of free space
$\epsilon_r =$ the relative permittivity or dielectric constant of the medium, having a value of unity for free space and essentially unity for air
$g =$ the conductivity of the medium expressed in mho-m/sq m, the numerical value for hard-drawn copper being 0.565×10^8 mho-m/sq m.

The symbol μ_0 may actually be interpreted to mean $\mu_0\mu_r$ where μ_r is the relative permeability of a ferromagnetic medium, but, since only nonferromagnetic materials are encountered here, the numeric μ_r is considered to be unity and as such does not appear in the equations.

Derivation of Maxwell's Equations in Cylindrical Coordinates. The derivation of Maxwell's equations in cylindrical coordinates follows exactly the general principle used in the above development for Cartesian coordinates, the differences having to do with the fact that some of the

elemental lengths are not independent of the other coordinates. The elemental volume to be employed here is shown in Fig. A–10 where the coordinates are r, θ, and z. It will be noticed in particular that the elemental length oa is a function of r as well as of θ.

(a) *Maxwell's First Law.* First the line-integral law of magnetism will be applied to the face *oabco* of the elemental volume at which the

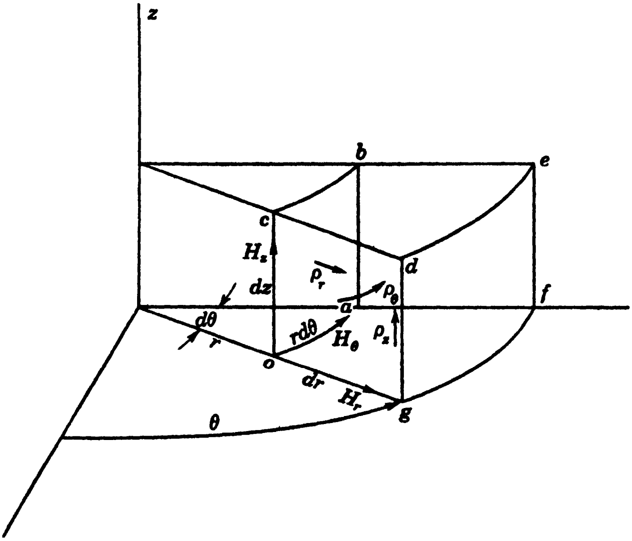

FIG. A–10. Element of volume in the electromagnetic field. Cylindrical coordinates.

current density ρ_r exists. The magnetic potential drops around the *oabco* path taken individually are

$$dU_{1(\text{along } oa)} = H_\theta r d\theta$$

$$dU_{2(\text{along } ab)} = H_z dz + \frac{\partial(H_z dz)}{\partial\theta} d\theta = H_z dz + \frac{1}{r}\cdot\frac{\partial H_z}{\partial\theta} r d\theta dz$$

(since dz is not a function of θ)

$$dU_{3(\text{along } bc)} = -\left[H_\theta r d\theta + \frac{\partial(H_\theta r d\theta)}{\partial z} dz\right] = -H_\theta r d\theta - \frac{\partial H_\theta}{\partial z} r d\theta dz$$

$$dU_{4(\text{along } co)} = -H_z dz$$

Hence

$$\oint H \cdot dl = dU_1 + dU_2 + dU_3 + dU_4 = \left(\frac{1}{r}\cdot\frac{\partial H_z}{\partial\theta} - \frac{\partial H_\theta}{\partial z}\right) r d\theta dz$$

$$= \rho_r r d\theta dz \qquad \text{[A–65]}$$

and

$$\frac{1}{r}\cdot\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = \rho_r \qquad \text{[A-66]}$$

or

$$\frac{1}{r}\cdot\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = gE_r + \varepsilon\frac{\partial E_r}{\partial t} \qquad \text{[A-67]}$$

A similar procedure may be applied to the other faces of the elemental volume which touch the point *o*, the point in space which is actually under investigation in the limit of shrinking dr, $d\theta$, and dz to zero. Because of the nonsymmetry of the coordinates however, it may be well to outline the method employed in applying the line integral law to the faces *ocdgo* and *ogfao*.

Consider first the *ocdgo* face across which the current I_θ $(= \rho_\theta drdz)$ flows. The individual magnetic potential drops around the periphery of this face are

$$dU_{1(\text{along } oc)} = H_z dz$$

$$dU_{2(\text{along } cd)} = H_r dr + \frac{\partial(H_r dr)}{\partial z} dz = H_r dr + \frac{\partial H_r}{\partial z} drdz$$

$$dU_{3(\text{along } dg)} = -\left[H_z dz + \frac{\partial(H_z dz)}{\partial r} dr\right] = -H_z dz - \frac{\partial H_z}{\partial r} drdz$$

$$dU_{4(\text{along } go)} = -H_r dr$$

Hence

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = gE_\theta + \varepsilon\frac{\partial E_\theta}{\partial t} \qquad \text{[A-68]}$$

Consider next the *ogfao* face across which the current I_z $(= \rho_z r d\theta dr)$ flows. The component magnetic potential drops taken around the periphery of this face are

$$dU_{1(\text{along } og)} = H_r dr$$

$$dU_{2(\text{along } gf)} = H_\theta r d\theta + \frac{\partial(H_\theta r d\theta)}{\partial r} dr = H_\theta r d\theta + \frac{1}{r}\cdot\frac{\partial(H_\theta r)}{\partial r} r d\theta dr$$

$$dU_{3(\text{along } fa)} = -\left[H_r dr + \frac{\partial(H_r dr)}{\partial \theta} d\theta\right] = -H_r dr - \frac{1}{r}\cdot\frac{\partial H_r}{\partial \theta} r d\theta dr.$$

$$dU_{4(\text{along } ao)} = -H_\theta r d\theta$$

It follows, as before, that

$$\frac{1}{r}\left[\frac{\partial(H_\theta r)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right] = gE_z + \varepsilon\frac{\partial E_z}{\partial t} \qquad \text{[A-69]}$$

Taken collectively, equations A–67, A–68, and A–69 describe the same physical facts in cylindrical coordinates as equations A–57, A–58, and A–59 do in rectangular coordinates.

(*b*) *Maxwell's Second Law.* In Fig. A–10 we may replace the ρ's by B's and apply Faraday's emf law to each of the faces of the elemental volume which touch point *o*. The magnetic flux piercing face *oabco*, for example, is $\phi_r = B_r r d\theta dz$, and the resulting induced emf is

$$e = -\oint \bar{E} \cdot d\bar{l} = -\frac{\partial B_r}{\partial t} r d\theta dz = -\mu_0 \frac{\partial H_r}{\partial t} r d\theta dz$$

Since $\bar{E}$ enters into the line integral exactly as $\bar{H}$ did previously, we may write, from equation A–67,

$$\frac{1}{r} \cdot \frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\mu_0 \frac{\partial H_r}{\partial t} \qquad \text{[A–70]}$$

From equation A–68,

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\mu_0 \frac{\partial H_\theta}{\partial t} \qquad \text{[A–71]}$$

From equation A–69,

$$\frac{1}{r}\left[\frac{\partial (E_\theta r)}{\partial r} - \frac{\partial E_r}{\partial \theta}\right] = -\mu_0 \frac{\partial H_z}{\partial t} \qquad \text{[A–72]}$$

Taken collectively, equations A–70, A–71, and A–72 specify in cylindrical coordinates the same interrelationships which exist between the E's and H's as equations A–62, A–63, and A–64 do in rectangular coordinates.

Illustrative Example. In order to illustrate the application of Maxwell's equations to a relatively simple case, the depth of penetration of an electromagnetic wave into a metal will be found employing the field configuration shown in Fig. A–11. The assumption will be made that at the surface of the conducting plate shown

$$E_x = E_z = 0 \quad \text{and} \quad H_y = H_z = 0 \qquad \text{[A–73]}$$

leaving only E_y and H_x as finite quantities. In accordance with Poynting's theorem,[2] it is understood that, with E_y and H_x present at the surface of metal, power will be crossing the surface and dissipated in the metal. Physically, E_y represents the electric intensity vector caused by current flow in the y direction, and H_x is the magnetic-field intensity vector established by the same current.

[2] See Art. 104.

On substitution of the relations given in equation A–73 into equations A–57, A–58, A–59, A–62, A–63, and A–64 there are obtained two equations in E_y and H_x

$$\frac{\partial H_x}{\partial z} = gE_y + \frac{\partial \varepsilon E_y}{\partial t} \qquad \text{[A–74]}$$

and

$$\frac{\partial E_y}{\partial z} = \frac{\partial \mu H_x}{\partial t} \qquad \text{[A–75]}$$

where E_y and H_x are functions of both time and space.

Let $$H_x = \mathcal{H}_x \epsilon^{j\omega t} \quad \text{and} \quad E_y = \mathcal{E}_y \epsilon^{j\omega t} \qquad \text{[A–76]}$$

where $\mathcal{H}_x$ and $\mathcal{E}_y$ are functions of space only. See page 271.

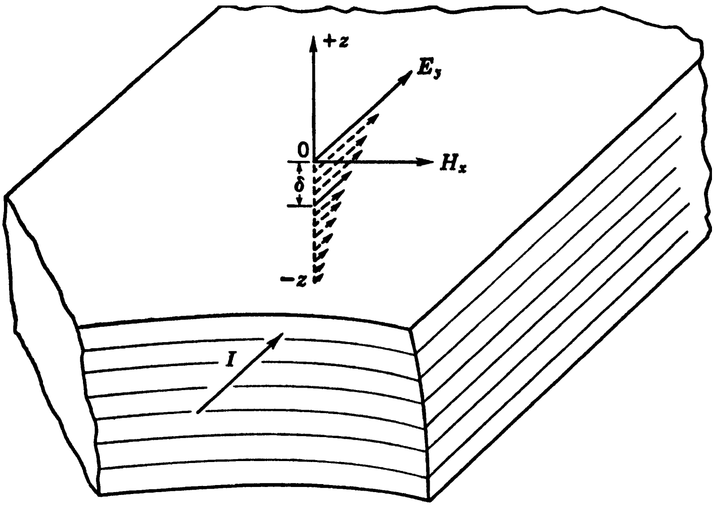

FIG. A–11. Illustrating depth of penetration of electromagnetic wave in metal.

Using the relations given by equation A–76, the time derivatives of H and E with respect to t become

$$\frac{\partial \mu H_x}{\partial t} = \frac{\partial \mu \mathcal{H}_x \epsilon^{j\omega t}}{\partial t} = j\omega\mu \mathcal{H}_x \epsilon^{j\omega t}$$

$$\frac{\partial \varepsilon E_y}{\partial t} = \frac{\partial \varepsilon \mathcal{E}_y \epsilon^{j\omega t}}{\partial t} \quad j\omega\varepsilon \mathcal{E}_y \epsilon^{j\omega t}$$

and then

$$\frac{\partial \mathcal{H}_x \epsilon^{j\omega t}}{\partial z} \quad (g + j\omega\varepsilon)\mathcal{E}_y \epsilon^{j\omega t} \qquad \text{[A–77]}$$

$$\frac{\partial \mathcal{E}_y \epsilon^{j\omega t}}{\partial z} \quad \cdot j\omega\mu \mathcal{H}_x \epsilon^{j\omega t} \qquad \text{[A–78]}$$

Dividing out the exponential terms gives

$$\frac{\partial \mathcal{H}_x}{\partial z} = (g + j\omega\varepsilon)\mathcal{E}_y \quad \text{[A-79]}$$

$$\frac{\partial \mathcal{E}_y}{\partial z} = j\omega\mu\mathcal{H}_x \quad \text{[A-80]}$$

Differentiate equation A-80, and substitute, from equation A-79,

$$\frac{\partial^2 \mathcal{E}_y}{\partial z^2} = j\omega\mu(g + j\omega\varepsilon)\mathcal{E}_y \quad \text{[A-81]}$$

In those cases where $g \gg \omega\varepsilon$, equation A-81 becomes[8]

$$\frac{\partial^2 \mathcal{E}_y}{\partial z^2} = j\omega\mu g\mathcal{E}_y \quad \text{[A-82]}$$

The solution of this type of differential equation is given by $\mathcal{E}_y = A\epsilon^{\gamma z}$ where A and γ are constants to be determined.

$$\frac{\partial \mathcal{E}_y}{\partial z} = \gamma A\epsilon^{\gamma z}$$

and

$$\frac{\partial^2 \mathcal{E}_y}{\partial z^2} = \gamma^2 A\epsilon^{\gamma z} = \gamma^2\mathcal{E}_y$$

Thus $\mathcal{E}_y = A\epsilon^{\gamma z}$ is a solution to equation A-82 provided

$$\gamma^2 = j\omega\mu g$$

or

$$\gamma = \sqrt{j\omega\mu g} = \sqrt{\frac{\omega\mu g}{2}} + j\sqrt{\frac{\omega\mu g}{2}} = s + js \quad \text{[A-83]}$$

Hence

$$E_y = \left[A\epsilon^{\sqrt{\frac{\omega\mu g}{2}}\cdot z}\right]\epsilon^{j\left(\omega t + \sqrt{\frac{\omega\mu g}{2}}\cdot z\right)} \qquad z \leqq 0 \quad \text{[A-84]}$$

wherein A is an arbitrary constant which is specified by the surface value of E_y at $z = 0$. The electric-field intensity vector E_y is thus shown to decrease exponentially for *negative* values of z as indicated in Fig. A-11.

The depth, along the $-z$ direction, at which $A\epsilon^{\sqrt{\frac{\omega\mu g}{2}}\cdot z}$ reduces to 0.368 (or ϵ^{-1}) of its surface value is called the depth of penetration. This depth, indicated as the distance δ in Fig. A-11, is

$$\frac{1}{\sqrt{\frac{\omega\mu g}{2}}} = \sqrt{\frac{\rho}{\pi f\mu}} = \sqrt{\frac{\rho}{\pi f\mu_0\mu_r}} \text{ meter} \quad \text{[A-85]}$$

[8] For copper conductors $g = 0.565 \times 10^8$ mho-m/sq m, and $\omega\varepsilon = 0.1745 \times 10^{-9}f$; hence for f equal to 10^9 cycles per second (1000 megacycles) $g \gg \omega\varepsilon$.

where $\rho = 1/g$ is the resistivity of the conductor material
$\mu_0 = 4\pi \times 10^{-7}$ henry/meter
$\mu_r =$ the relative permeability of the metal

giving

$$\delta = 503\sqrt{\frac{\rho}{f\mu_r}} \quad \text{meter} \qquad \text{[A-86]}$$

This particular depth of penetration is used to establish approximate formulae for the high-frequency resistance of metallic conductors. (See Art. 12.)

It is also of interest to calculate an effective or intrinsic impedance of the metal using expressions for E_y and H_x within it. The component E_y can be written from equation A-84 as

$$E_{y(\text{real})} = A\epsilon^{sz} \cos(\omega t + sz) \qquad z \leqq 0$$

or

$$E_{y(\text{real})} = A\epsilon^{sz} \cos(\omega t + sz)\ \underline{/0^\circ} \qquad \text{[A-87]}$$

Equation A-80 can be written as

$$\mathcal{H}_x = -\frac{j}{\omega\mu} \cdot \frac{\partial \mathcal{E}_y}{\partial z}$$

or

$$H_x = -\frac{j}{\omega\mu} \cdot \frac{\partial E_y}{dz} \qquad \text{[A-88]}$$

$$= -\frac{j}{\omega\mu}[A\epsilon^{sz} js\epsilon^{j(\omega t+sz)} + As\epsilon^{sz}\epsilon^{j(\omega t+sz)}]$$

$$= \frac{As}{\omega\mu}\epsilon^{sz}\epsilon^{j(\omega t+sz)}(1 - j)$$

$$= \frac{As}{\omega\mu}\epsilon^{sz}(1 - j)[\cos(\omega t + sz) + j\sin(\omega t + sz)]$$

or

$$H_{x(\text{real})} = \frac{\sqrt{2}\,As}{\omega\mu}\epsilon^{sz}\cos\left(\omega t + sz - \frac{\pi}{4}\right)$$

or

$$H_{x(\text{real})} = \frac{\sqrt{2}\,As}{\omega\mu}\epsilon^{sz}\cos(\omega t + sz)\ \underline{/-45^\circ} \qquad \text{[A-89]}$$

Using equations A-87 and A-89, an intrinsic impedance Z_i can be defined for the metal as

$$Z_i = \frac{E_y}{H_x} = \frac{\omega\mu}{\sqrt{2}s}\underline{/45^\circ} \quad \text{ohms} \qquad \text{[A-90]}$$

or on substituting for s, from equation A-83,

$$Z_i = \sqrt{\frac{\omega\mu}{g}}\ \underline{/45^\circ} \quad \text{ohms} \qquad \text{[A-91]}$$

which indicates that within the metal the magnetic field lags the electric field by 45°.

APPENDIX VII

ELEMENTS OF BESSEL FUNCTIONS

A. Solution of the Zero-Order Equation. The immediate problem is to obtain a solution of the following differential equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\cdot\frac{dy}{dx} + y = 0 \qquad \text{[A–92]}$$

This equation is " Bessel's equation of order zero " and is the simplest of the equations to be considered here. As a solution of equation A–92 try the series:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \qquad \text{[A–93]}$$

This may be written $y = a_nx^n$ if it is remembered that a summation is implied. First however let the first few terms of equation A–93 be substituted into equation A–92.

$$\frac{1}{x}\cdot\frac{dy}{dx} = \frac{1}{x}(a_1 + 2a_2x + 3a_3x^2 + \cdots)$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + \cdots$$

Thus, on substitution into equation A–92,

$$2a_2 + 6a_3x + \frac{a_1}{x} + 2a_2 + 3a_3x + a_0 + a_1x + (\text{higher powers of } x) = 0$$

This expression, instead of being zero when $x = 0$, becomes infinite, owing to the term a_1/x. Thus a_1 must be set equal to zero. Remembering that $a_1 = 0$ and going back to the solution

$$y = a_nx^n$$

$$\frac{dy}{dx} = na_nx^{n-1}$$

$$\frac{d^2y}{dx^2} = n(n-1)a_nx^{n-2}$$

equation A–92 becomes

$$n(n-1)a_nx^{n-2} + na_nx^{n-2} + a_nx^n = 0 \qquad \text{[A–94]}$$

The left side of this equation represents a polynomial with an infinite number of terms, each made up of a coefficient and some power of x. Since the equality must hold for all values of x, then the coefficient of each power of x must be zero. In equation A–94 two powers of x are represented, x^n and x^{n-2}. A selection must be made so that only one power is given. This can be done by substituting for all n's in the first two terms the quantity $(n + 2)$. In order to see this clearly it may be necessary to write out the first few terms of the series. When this is done there results

$$(n + 2)(n + 1)a_{n+2}x^n + (n + 2)a_{n+2}x^n + a_nx^n = 0$$

The x^n can now be factored out and the coefficient set equal to zero. Thus

$$a_{n+2}[(n + 2)(n + 1) + (n + 2)] + a_n = 0$$

Solving for a_{n+2} gives

$$a_{n+2} = \frac{-a_n}{(n + 2)^2} \qquad \text{[A–95]}$$

All the coefficients of the series can be found from this equation. It is found that two of the coefficients can be arbitrarily chosen. One of these, a_1, has already been set equal to zero; the other, a_0, can be set equal to unity. On this basis then the coefficients are

$$a_0 = 1$$

$$a_1 = 0$$

From equation A–95,

$$a_2 = -\frac{1}{2^2}$$

$$a_3 = 0$$

$$a_4 = -\frac{a_2}{4^2} = \frac{1}{2^2 \cdot 4^2}$$

$$a_5 = 0$$

$$a_6 = -\frac{1}{2^2 \cdot 4^2 \cdot 6^2}$$

$$\cdots$$

The series A–93 now becomes

$$y = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots = J_0(x) \qquad \text{[A–96]}$$

This is one particular solution of the equation A–92. It is usually written as $J_0(x)$ and called *Bessel's function of the first kind and zero order*. At $x = 0$ it is seen to have the following characteristics:

$$J_0(x) = y = 1 \qquad\qquad x = 0$$

$$\frac{dJ_0(x)}{dx} = \frac{dy}{dx} = 0 \qquad\qquad x = 0$$

Since equation A–92 is a second-order differential equation, its general solution must be made up of a combination of two independent particular solutions, one of which is $J_0(x)$. The other standard particular solution is denoted by $Y_0(x)$ and is called *Bessel's function of the second kind and zero order*. This is the form standardized by Weber, and its derivation is beyond the scope of this brief treatment.[1] Values of both $J_0(x)$ and $Y_0(x)$ can be found in suitable tables.[2] Curves of $J_0(x)$ and $Y_0(x)$ are given in Figs. A–12 and A–13 respectively. The general solution of equation A–92 then becomes

$$y = AJ_0(x) + BY_0(x) \qquad \text{[A–97]}$$

It will be of interest later to write out the first derivative of $J_0(x)$. It is as follows (from equation A–96):

$$\frac{dJ_0(x)}{dx} = -\frac{x}{2} + \frac{x^3}{2^2 \cdot 4} - \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \cdots$$

$$= -x\left[\frac{1}{2} - \frac{x^2}{2^2 \cdot 4} + \frac{x^4}{2^2 \cdot 4^2 \cdot 6} - \cdots\right] \qquad \text{[A–98]}$$

B. Solution of the First-Order Equation. Bessel's equation of the first order is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\cdot\frac{dy}{dx} + \left(1 - \frac{1}{x^2}\right)y = 0 \qquad \text{[A–99]}$$

Let a solution of this equation be assumed in the form

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots) \qquad \text{[A–100]}$$

where the a's are not necessarily the same as in the previous article.

[1] Refer, for a further treatment, to *Mathematical Methods in Engineering*, by Kármán, T. and M. A. Biot, p. 50, McGraw-Hill Book Co., New York, 1940.

[2] Jahnke, E., and F. Emde, *Funktionentafeln*, Leipzig, B. G. Teubner, 1938. McLachlan, N. W., *Bessel Functions for Engineers*, New York, Oxford University Press, 1934. Watson, G. N , *A Treatise on the Theory of Bessel Functions*, Cambridge (Eng.), The University Press, 1922.

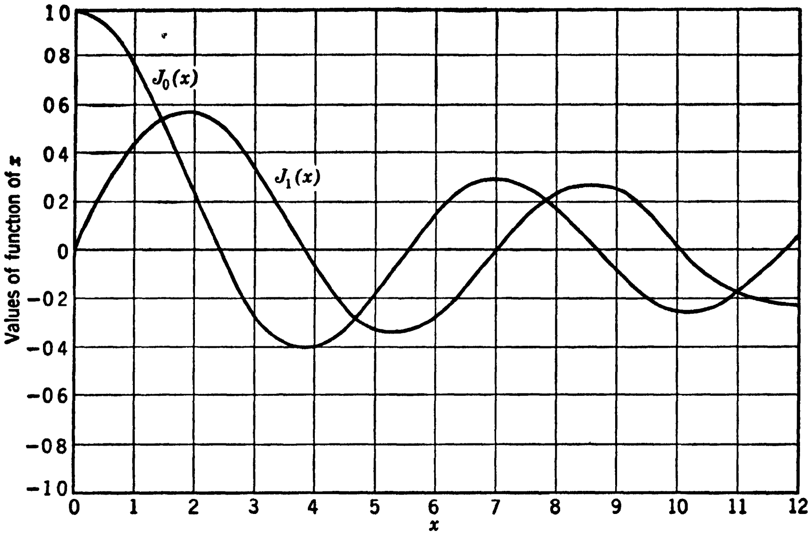

FIG A–12 Curves representing the variation of $J_0(x)$ and $J_1(x)$ with x

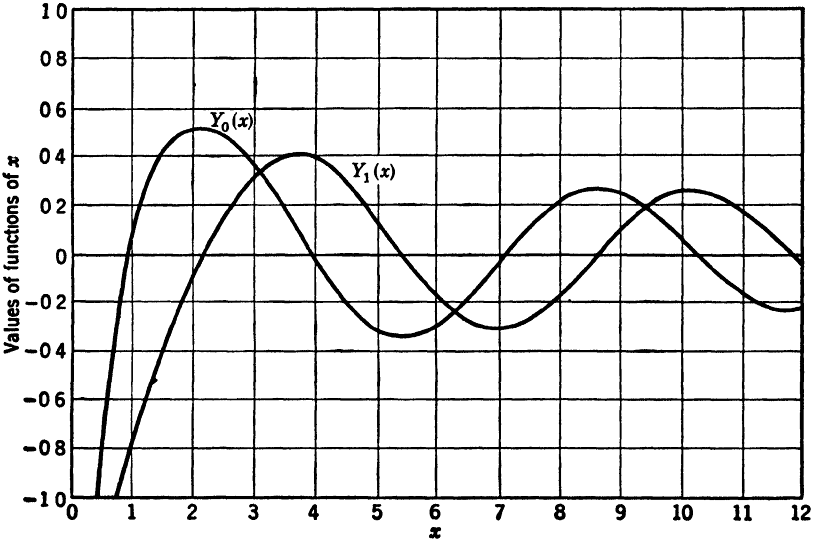

FIG. A–13. Curves representing the variation of $Y_0(x)$ and $Y_1(x)$ with x.

Following a procedure somewhat like the foregoing,

$$y = a_n x^{n+m}$$

$$\left(1 - \frac{1}{x^2}\right) a_n x^{n+m} = a_n x^{n+m} - a_n x^{n-2+m}$$

$$\frac{1}{x} \cdot \frac{dy}{dx} = \frac{1}{x}(n + m)a_n x^{n+m-1} = (n + m)a_n x^{n+m-2}$$

$$\frac{d^2y}{dx^2} = (n + m)(n + m - 1)a_n x^{n+m-2}$$

Thus equation A–99 becomes after the substitution

$$(n + m)(n + m - 1)a_n x^{n+m-2} + (n + m)a_n x^{n+m-2} + a_n x^{n+m} - a_n x^{n+m-2} = 0$$

Factor out x^m.

$$x^m[(n + m)(n + m - 1)a_n x^{n-2} + (n + m)a_n x^{n-2} + a_n x^n - a_n x^{n-2}] = 0 \qquad \text{[A–101]}$$

Remembering that the lowest value of n is zero, write out the coefficient of x^{-2}. This is, letting $n = 0$ in the first, second, and fourth terms,

$$m(m - 1)a_0 + ma_0 - a_0 = 0$$

from which $m = 1$.

Write out the coefficient of x^{-1} by using $n = 1$ in the first, second, and fourth terms.

$$(m + 1)ma_1 + (m + 1)a_1 - a_1 = 0$$

where from the above, $m = 1$. Therefore

$$2a_1 + 2a_1 - a_1 = 0$$

which requires a_1 to be zero.

Going back to equation A–101 and writing out the general coefficient by substituting for n in the first, second, and fourth terms the quantity $(n + 2)$; and factoring out x^m and x^n:

$$(n + m + 2)(n + m + 1)a_{n+2} + (n + m + 2)a_{n+2} + a_n - a_{n+2} = 0$$

where again $m = 1$. Therefore

$$(n + 3)(n + 2)a_{n+2} + (n + 3)a_{n+2} + a_n - a_{n+2} = 0$$

Solve for a_{n+2}.

$$a_{n+2} = \frac{-a_n}{(n + 4)(n + 2)} \qquad \text{[A–102]}$$

The coefficients for the series become

$$a_0 = a_0$$

$$a_1 = 0$$

From equation A–102,

$$a_2 = -\frac{a_0}{4 \cdot 2}$$

$$a_3 = 0$$

$$a_4 = \frac{a_0}{6 \cdot 4 \cdot 4 \cdot 2}$$

$$\cdots$$

Thus the solution becomes

$$y = x\left(a_0 - \frac{a_0 x^2}{4 \cdot 2} + \frac{a_0 x^4}{6 \cdot 4 \cdot 4 \cdot 2} - \cdots\right)$$

In this solution a_0 is arbitrary and is set equal to $\frac{1}{2}$ in the standard form denoted by $J_1(x)$:

$$J_1(x) = x\left(\frac{1}{2} - \frac{x^2}{2^2 \cdot 4} + \frac{x^4}{2^2 \cdot 4^2 \cdot 6} - \cdots\right)$$

which is seen to be the same as equation A–98 except for sign. Thus in general the first derivative of the solution of the zero-order equation is the negative of the solution for the first-order equation. That is

$$\frac{dJ_0(x)}{dx} = -J_1(x) \qquad \text{[A–103]}$$

There is also a solution of the second kind for the first-order equation, corresponding to $Y_0(x)$. It is

$$Y_1(x) = -\frac{dY_0(x)}{dx} \qquad \text{[A–104]}$$

For Bessel's first-order equation the general solution is

$$y = CJ_1(x) + DY_1(x) \qquad \text{[A–105]}$$

where $J_1(x)$ and $Y_1(x)$ can be found in appropriate tables, for usual values of x. Curves of $J_1(x)$ and $Y_1(x)$ are found in Figs. A–12 and A–13 respectively.

C. The Hankel Functions of Zero Order. It has been seen that two solutions for the equation of zero order are $J_0(x)$ and $Y_0(x)$. Various combinations of these two functions may be used as solutions of Bessel's

equation, depending on circumstances. Two of these combinations are immediately useful in transmission theory and are defined as

$$H_0^1(x) = J_0(x) + jY_0(x) \qquad \text{[A-106]}$$

and

$$H_0^2(x) = J_0(x) - jY_0(x) \qquad \text{[A-107]}$$

These are known as the *Bessel functions of the third kind and zero order* or as *Hankel functions of zero order.*

D. Transformation of Variable. A differential equation frequently appears as follows:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\cdot\frac{dy}{dx} + m^2y = 0 \qquad \text{[A-108]}$$

where m is a constant. This equation can be transformed into the standard Bessel equation of zero order by a change of variable, $z = mx$.

$$dx = \frac{dz}{m}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{md}{dz}\left(\frac{mdy}{dz}\right) = m^2\frac{d^2y}{dz^2}$$

Substituting into equation A-108

$$m^2\frac{d^2y}{dz^2} + \frac{m^2}{z}\cdot\frac{dy}{dz} + m^2y = 0$$

out of which the m's can be canceled, resulting in equation A-92, for which solutions have been found to be $J_0(z)$ and $Y_0(z)$. Thus the solution for equation A-108 is

$$y = AJ_0(mx) + BY_0(mx) \qquad \text{[A-109]}$$

Similarly the solution for the first-order equation can be found to be

$$y = CJ_1(mx) + DY_1(mx) \qquad \text{[A-110]}$$

APPENDIX VIII

WIRE TABLE, STANDARD ANNEALED AND HARD DRAWN (97.3% Cond.) COPPER

Gage No. A.W.G.	Diameter in Mils at 20°C	Cross Section at 20°C		Ohms per 1000 feet*				
		Circular mils	Square inches	Annealed 20°C, 68°F	Hard Drawn (97.3% Conductivity) 0°C, 32°F	20°C, 68°F	25°C, 77°F	50°C, 122° F
0000	460.0	211 600.	0.1662	0.049 01	0.046 52	0.050 37	0.051 33	0.056 15
000	409.6	167 800.	0.1318	0.061 80	0.058 67	0.063 52	0.064 73	0.070 80
00	364.8	133 100.	0.1045	0.077 93	0.073 98	0.080 10	0.081 62	0.089 27
0	324.9	105 500.	0.082 89	0.098 27	0.093 28	0.101 0	0.102 9	0.112 6
1	289.3	83 690.	0.065 73	0.123 9	0.117 6	0.127 4	0.129 8	0.142 0
2	257.6	66 370.	0.052 13	0.156 3	0.148 3	0.160 6	0.163 7	0.179 0
3	229.4	52 640.	0.041 34	0.197 0	0.187 0	0.202 5	0.206 4	0.225 7
4	204.3	41 740.	0.032 78	0.248 5	0.235 8	0.255 4	0.260 2	0.284 6
5	181.9	33 100.	0.026 00	0.313 3	0.297 4	0.322 0	0.328 1	0.358 9
6	162.0	26 250.	0.020 62	0.395 1	0.375 0	0.406 0	0.413 8	0.452 6
7	144.3	20 820.	0.016 35	0.498 2	0.472 9	0.512 0	0.521 8	0.570 7
8	128.5	16 510.	0.012 97	0.628 2	0.596 3	0.645 6	0.657 9	0.719 6
9	114.4	13 090.	0.010 28	0.792 1	0.751 9	0.814 1	0.829 7	0.907 4
10	101.9	10 380.	0.008 155	0.998 9	0.948 1	1.027	1.046	1.144
11	90.74	8234.	0.006 467	1.260	1.196	1.294	1.319	1.443
12	80.81	6530.	0.005 129	1.588	1.508	1.632	1.663	1.819
13	71.96	5178.	0.004 067	2.003	1.901	2.058	2.098	2.294
14	64.08	4107.	0.003 225	2.525	2.397	2.595	2.645	2.893
15	57.07	3257.	0.002 558	3.184	3.023	3.273	3.335	3.648
16	50.82	2583.	0.002 028	4.016	3.812	4.127	4.206	4.600
17	45.26	2048.	0.001 609	5.064	4.806	5.204	5.303	5.800
18	40.30	1624.	0.001 276	6.385	6.061	6.562	6.687	7.314
19	35.89	1288.	0.001 012	8.051	7.642	8.275	8.433	9.223
20	31.96	1022.	0.000 802 3	10.15	9.637	10.43	10.63	11.63
21	28.46	810.1	0.000 636 3	12.80	12.15	13.16	13.41	14.66
22	25.35	642.4	0.000 504 6	16.14	15.32	16.59	16.91	18.49
23	22.57	509.5	0.000 400 2	20.36	19.32	20.92	21.32	23.32
24	20.10	404.0	0.000 317 3	25.67	24.37	26.38	26.88	29.40
25	17.90	320.4	0.000 251 7	32.37	30.72	33.27	33.90	37.08
26	15.94	254.1	0.000 199 6	40.81	38.74	41.95	42.75	46.75
27	14.20	201.5	0.000 158 3	51.47	48.85	52.89	53.90	58.96
28	12.64	159.8	0.000 125 5	64.90	61.60	66.70	67.97	74.34
29	11.26	126.7	0.000 099 53	81.83	77.68	84.10	85.71	93.74
30	10.03	100.5	0.000 078 94	103.2	97.95	106.1	108.1	118.2
31	8.928	79.70	0.000 062 60	130.1	123.5	133.7	136.3	149.1
32	7.950	63.21	0.000 049 64	164.1	155.7	168.6	171.9	188.0
33	7.080	50.13	0.000 039 37	206.9	196.4	212 6	216.7	237.0
34	6.305	39.75	0.000 031 22	260.9	247.6	268.1	273.3	298.9
35	5.615	31.52	0.000 024 76	329.0	312.3	338.1	344.6	376.9
36	5.000	25.00	0.000 019 64	414.8	393.8	426.4	434.5	475.2
37	4.453	19.83	0.000 015 57	523.1	496.5	537.6	547.9	599.2
38	3.965	15.72	0.000 012 35	659.6	626.1	677.9	690.9	755.6
39	3.531	12.47	0.000 009 793	831.8	789.5	854.8	871.2	952.8
40	3.145	9.888	0.000 007 766	1,049.0	995.6	1,078.	1,099.	1,201.

*Resistance at the stated temperatures of a wire whose length is 1000 feet at 20°C.

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